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Endochronic theory, non-linear kinematic hardening rule and generalized plasticity: a new interpretation based on generalized normality assumption

Silvano Erlicher [a,*], Nelly Point [a,b]

[a] *Laboratoire d'Analyse des Matériaux et Identification LAMI (ENPC/LCPC-Institut Navier) 6 et 8 av. B. Pascal, Cité Descartes, Champs-sur-Marne, 77455 Marne-la-Vallée, Cedex 2, France*
[b] *Conservatoire National des Arts et Métiers CNAM, Spécialité Mathématiques (442), 292 rue Saint-Martin, 75141 Paris, Cedex 03, France*

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Abstract

A simple way to define the flow rules of plasticity models is the assumption of generalized normality associated with a suited pseudo-potential function. This approach, however, is not usually employed to formulate endochronic theory and non-linear kinematic (NLK) hardening rules as well as generalized plasticity models. In this paper, generalized normality is used to give a new formulation of these classes of models. As a result, a suited pseudo-potential is introduced for endochronic models and a non-standard description of NLK hardening and generalized plasticity models is also provided. This new formulation allows for an effective investigation of the relationships between these three classes of plasticity models.

Keywords: Thermodynamics of solids; Plasticity; Generalized standard materials; Nonlinear kinematic hardening; Endochronic theory; Generalized plasticity

1. Introduction

The models proposed so far in the literature to describe the rate independent inelastic behavior of real materials subjected to monotonic or cyclic loading conditions can be essentially classified into two main

* Corresponding author. Tel.: +33 1 64 15 37 80; fax: +33 1 64 15 37 41.
E-mail address: erlicher@lami.enpc.fr (S. Erlicher).

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families: (i) models where the present state depends on the present value of observable variables (total strain, temperature) and of suitable internal variables; (ii) models, indicated here as *hereditary*, that require the knowledge of the whole past history of observable variables.

The first group encompasses, for instance, the classical models of Prandtl–Reuss and Prager (see e.g. Lemaitre and Chaboche, 1990) and the NLK hardening model of Armstrong and Frederick (1966), in its original form as well as in the modified versions recently proposed by Chaboche (1991) and Ohno and Wang (1993) in order to improve the ratchetting modelling. For these models, the well known notions of *elastic domain* and *loading* (or *yielding*) *surface* apply. *Associativity* and *non-associativity* of the plastic strain flow rule are also well-established concepts, as well as the assumption of generalized associativity (or *generalized normality*), relating all internal variable flow directions to a given loading surface (Halphen and Nguyen, 1975; Jirásek and Bažant, 2002). Using the language of convex analysis (Rockafellar, 1969), generalized normality entails that the flows of all internal variables belong to the *sub-differential set* of a given scalar non-negative function called *pseudo-potential* (Moreau, 1970; Frémond, 2002).

Among internal variable theories, *generalized plasticity* deserves special attention. A first important step for its formulation was the idea, suggested by Eisenberg and Phillips (1971), of a plasticity model where, despite classical plasticity, loading and yielding surfaces are *not* coincident. Then, starting from an axiomatic approach to describe inelastic behavior of materials, Lubliner proposed some simple generalized plasticity models, able to represent some observed experimental behavior of metals (Lubliner, 1974, 1980, 1984; Lubliner et al., 1993). More recently, generalized plasticity has been used for describing the shape memory alloy behavior (Lubliner and Auricchio, 1996).

Endochronic models (Valanis, 1971) and *Bouc-Wen* type models (Bouc, 1971; Wen, 1976) are two important examples of hereditary models. Endochronic theory has been developed during the seventies and used for modelling the plastic behavior of metals (see, for instance, Valanis, 1971; Valanis and Wu, 1975) and the inelastic behavior of concrete and soils (among others, Bažant and Krizek, 1976; Bažant and Bath, 1976). The endochronic stress evolution rule depends on the so-called *intrinsic time* and is formulated by a convolution integral between the strain tensor and a scalar function of the intrinsic time called *memory kernel*. When the kernel is an exponential function, an incremental form of endochronic flow rules exists, which is commonly used in standard analyses and applications.

Models of Bouc-Wen type are widely employed for modelling the cyclic behavior of structures in seismic engineering (Baber and Wen, 1981; Sivaselvan and Reinhorn, 2000) and for representing hysteresis of magneto-rheological dampers in semi-active control applications (Sain et al., 1997; Jansen and Dyke, 2000). The strict relationship between endochronic and Bouc-Wen type models has been mentioned several times in the literature (see, among others, Karray and Bouc, 1989; Casciati, 1989). Recently, Erlicher and Point (2004) showed that the fundamental element of this relationship is the choice of an appropriate intrinsic time.

Endochronic theory and classical internal variable theory have been compared by using several approaches: Bažant (1978) observed that for endochronic theory the notion of loading surface can still be introduced, but it looses its physical meaning; Valanis (1980) and Watanabe and Atluri (1986) proved that a NLK hardening model can be derived from an endochronic model by imposing a special intrinsic time definition. Moreover, a comparative study between NLK hardening and generalized plasticity models has been presented by Auricchio and Taylor (1995). A tight relationship between endochronic theory and generalized plasticity is also expected to exist, but, by the authors' knowledge, no analysis on this subject has been done. More generally, there is a lack of unified theoretical framework, on which formal comparisons between these plasticity theories could be based. The main goal of this paper is the formulation of this theoretical framework using the classical notion of *generalized normality* (Moreau, 1970; Halphen and Nguyen, 1975). As a result, a new formulation of endochronic and NLK hardening models as well as generalized plasticity models is suggested and is used to investigate the relationships between them.

The paper is organized as follows: in the first section, the standard theoretical framework of thermomechanics is briefly recalled, with reference to the notions of pseudo-potential and generalized normality as well as Legendre–Fenchel transform and dual pseudo-potential. In the following sections, several plasticity models are presented and are shown to fulfil the generalized normality assumption. The Prandtl–Reuss and endochronic models are considered first, in both standard and multi-layer formulations. Then, NLK hardening model and generalized plasticity follow. The discussion is limited to initially isotropic materials, whose plastic behavior is governed by the second invariant of the deviatoric stress, J_2, known as von Mises or J_2 materials. No stability analysis is provided, as it is beyond the purposes of this contribution.

2. General thermodynamic framework

Under the assumption of infinitesimal transformations, the classical expression of the local form of the first and second principle of thermodynamics can be written as follows:

$$T\dot{s} = -\dot{\Psi} - s\dot{T} - \operatorname{div}(\mathbf{q}) + r + \boldsymbol{\sigma}\colon \dot{\boldsymbol{\varepsilon}} \tag{1}$$

$$\Phi_s(t) = \dot{s} + \operatorname{div}\left(\frac{\mathbf{q}}{T}\right) - \frac{r}{T} \geqslant 0 \tag{2}$$

where the superposed dot indicates the time derivative; s is the entropy density per unit volume, $\mathbf{q}$ is the vector of the flowing out heat flux, T is the absolute temperature, r is the rate of heat received by the unit volume of the system from the exterior; $\boldsymbol{\sigma}$ is the second order symmetric Cauchy stress tensor; $\boldsymbol{\varepsilon}$ is the tensor of small total strains; $\Phi_s(t)$ is the rate of interior entropy production. In the vector space of all second order tensors, the Euclidean scalar product : is defined by $\mathbf{x}{:}\mathbf{y} = x_{ij}y_{ij}$; the vector subspace of second order symmetric tensors is denoted by $\mathbb{S}^2$. The Helmholtz free energy density per unit volume is a state function defined as

$$\Psi = \Psi(\boldsymbol{\varepsilon}, T, \boldsymbol{\chi}_1, \ldots, \boldsymbol{\chi}_N) = \Psi(\mathbf{v}) \tag{3}$$

where $\boldsymbol{\chi}_1, \ldots, \boldsymbol{\chi}_N$, are the tensorial and/or scalar internal variables, related to the non-elastic evolution and $\mathbf{v} = \{\boldsymbol{\varepsilon}, T, \boldsymbol{\chi}_1, \ldots, \boldsymbol{\chi}_N\}$ is the vector containing all the state variables, namely the total strain tensor, the temperature and the internal variables.

For isothermal conditions, the use of Eq. (1) in the inequality (2) leads to

$$\Phi_m(t){:} = T\Phi_s(t) = T\dot{s} = \boldsymbol{\sigma}\colon \dot{\boldsymbol{\varepsilon}} - \dot{\Psi} \geqslant 0 \tag{4}$$

which states that the *intrinsic* or *mechanical dissipation* Φ_m (rate of energy per unit volume) must be nonnegative. The *non-dissipative* thermodynamic forces are defined as functions of the free energy density Ψ (see, among others, Frémond, 2002)

$$\boldsymbol{\sigma}^{nd} := \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}}, \quad \boldsymbol{\tau}_i^{nd} := \frac{\partial \Psi}{\partial \boldsymbol{\chi}_i} \Longleftrightarrow \mathbf{q}^{nd} = \frac{\partial \Psi}{\partial \mathbf{v}} \tag{5}$$

The non-dissipative stress $\boldsymbol{\sigma}^{nd}$ is associated with the observable variable $\boldsymbol{\varepsilon}$, while $\boldsymbol{\tau}_i^{nd}$ are associated with the internal variables $\boldsymbol{\chi}_i$. All non-dissipative forces can be collected in $\mathbf{q}^{nd} = (\boldsymbol{\sigma}^{nd}, \boldsymbol{\tau}_1^{nd}, \ldots, \boldsymbol{\tau}_N^{nd})$. Hence, by substituting (5) into (4), one obtains

$$\Phi_m(t) = (\boldsymbol{\sigma} - \boldsymbol{\sigma}^{nd})\colon \dot{\boldsymbol{\varepsilon}} - \sum_{i=1}^{N} \boldsymbol{\tau}_i^{nd} \cdot \dot{\boldsymbol{\chi}}_i = \boldsymbol{\sigma}\colon \dot{\boldsymbol{\varepsilon}} - q^{nd} \cdot \dot{\mathbf{v}} \geqslant 0 \tag{6}$$

where $\dot{\mathbf{v}}$ is the vector of the fluxes, belonging to a suitable vector space $\mathbb{V}$. The vector spaces considered in this paper are isomorph to $\mathbb{R}^n$ and the same hold for their dual $\mathbb{V}^*$ (see Appendix A). The symbol $\cdot$ indicates the scalar product of two objects having the same structure: two tensors, two scalar variables or two collections of tensorial and/or scalar variables (the same notation has been used e.g. by Jirásek and Bažant (2002, p. 428)). The inequality (6) can be written in a slightly different manner, by introducing the *dissipative* thermodynamic forces

$$\mathbf{q}^d = [\boldsymbol{\sigma}^d, \boldsymbol{\tau}_1^d, \ldots, \boldsymbol{\tau}_N^d] := [\boldsymbol{\sigma} - \boldsymbol{\sigma}^{nd}, -\boldsymbol{\tau}_1^{nd}, \ldots, -\boldsymbol{\tau}_N^{nd}] \in \mathbb{V}^* \tag{7}$$

Hence,

$$\Phi_m(t) = \boldsymbol{\sigma}^d : \dot{\boldsymbol{\varepsilon}} + \sum_{i=1}^{N} \boldsymbol{\tau}_i^d \cdot \dot{\boldsymbol{\chi}}_i = \mathbf{q}^d \cdot \dot{\mathbf{v}} \geqslant 0 \tag{8}$$

The forces $\mathbf{q}^d$ have to be defined in such a way that the couple $(\mathbf{q}^d, \dot{\mathbf{v}})$ always fulfils the inequality (8). Therefore, some additional *complementary rules* have to be introduced. They can be defined by assuming the existence of a non-negative, proper, convex and lower semi-continuous function $\phi : \mathbb{V} \to (-\infty, \infty]$ (Appendix A, item 2), called *pseudo-potential*, in general non-differentiable, such that $\phi(\mathbf{0}) = 0$ and:

$$\mathbf{q}^d \in \partial\phi(\dot{\mathbf{v}}) \tag{9}$$

where ∂ indicates the sub-differential operator (see Appendix, item 3). This condition is called *generalized normality*. A more detailed way of writing (9) is

$$\mathbf{q}^d \in \partial\phi(\dot{\mathbf{v}}';\mathbf{v})|_{\dot{\mathbf{v}}'=\dot{\mathbf{v}}} \tag{10}$$

As a matter of fact, ϕ is a general function of the fluxes $\dot{\mathbf{v}}'$ and may also depend on the state variables $\mathbf{v}$. However, the subdifferential is taken, by definition, only with respect to the fluxes $\dot{\mathbf{v}}'$ and the thermodynamic force $\mathbf{q}^d$ corresponds to the subdifferential of ϕ at $\dot{\mathbf{v}}' = \dot{\mathbf{v}}$, where $\dot{\mathbf{v}}$ is the *actual* flow. By using the properties of sub-differentials, it can be proved that for dissipative forces defined by (9), the inequality $\mathbf{q}^d \cdot \dot{\mathbf{v}} \geqslant 0$ is always fulfilled (Appendix A, item 4). Therefore, the second principle (8) is also satisfied.

A dual pseudo-potential $\phi^* : \mathbb{V}^* \to (-\infty, \infty]$ can be defined by the Legendre–Fenchel transform of ϕ:

$$\phi^*(\mathbf{q}^{d'}) := \sup_{\dot{\mathbf{v}}' \in \mathbb{V}}(\mathbf{q}^{d'} \cdot \dot{\mathbf{v}}' - \phi(\dot{\mathbf{v}}')) \tag{11}$$

When ϕ has an additional dependence on the state variables $\mathbf{v}$, then (11) leads to $\phi^* = \phi^*(\mathbf{q}^{d'};\mathbf{v})$. It can be proved that the dual pseudo-potential is a non-negative, proper, convex and lower semi-continuous function of $\mathbf{q}^{d'}$, such that $\phi^*(\mathbf{0}) = 0$ (see Appendix A, item 5). The dual normality condition reads

$$\dot{\mathbf{v}} \in \partial\phi^*(\mathbf{q}^d) \tag{12}$$

where $\mathbf{q}^d$ is the actual value of the dissipative force. The expression (12) is equivalent to

$$\dot{\mathbf{v}} \in \partial\phi^*(\mathbf{q}^{d'};\mathbf{v})|_{\mathbf{q}^{d'}=\mathbf{q}^d} \tag{13}$$

and it guarantees that $\mathbf{q}^d \cdot \dot{\mathbf{v}} \geqslant 0$ (Appendix A, item 5). Moreover, it defines the complementarity rules of *generalized standard materials* (Halphen and Nguyen, 1975), sometimes called *fully associated materials* (Jirásek and Bažant, 2002, p. 452).

Plasticity is characterized by a *rate-independent* memory effect (Visintin, 1994, p. 13). This special behavior occurs when the pseudo-potential ϕ is a positively homogeneous function of order 1 with respect to the fluxes $\dot{\mathbf{v}}'$. In this case, provided that $\mathbf{q}^d$ is computed from (9) or that $\dot{\mathbf{v}}$ derives from (12), it can be proved that the pseudo-potential at $\dot{\mathbf{v}}$ is equal to the intrinsic dissipation, viz. $\phi(\dot{\mathbf{v}}) = \mathbf{q}^d \cdot \dot{\mathbf{v}} = \Phi_m$ (Appendix A, item 6). Moreover, the dual pseudo-potential ϕ^* becomes the indicator function of a closed convex set $\overline{\mathbb{E}} \subset \mathbb{V}^*$ and the normality rule (12) entails that, given the dissipative force $\mathbf{q}^d \in \overline{\mathbb{E}}$, the flux $\dot{\mathbf{v}}$ fulfils the following condition:

$$\forall \mathbf{q}^{d'} \in \overline{\mathbb{E}} \quad (\mathbf{q}^{d'} - \mathbf{q}^d) \cdot \dot{\mathbf{v}} \leqslant 0 \tag{14}$$

viz. for a given dissipative force $\mathbf{q}^d$, the flow $\dot{\mathbf{v}}$ defined by (14) (or, equivalently, by (12) or (13)) is such that its power when it is associated to the actual force $\mathbf{q}^d$ is always greater or equal to the power $\mathbf{q}^{d'} \cdot \dot{\mathbf{v}}$ of all the other dissipative forces $\mathbf{q}^{d'} \in \overline{\mathbb{E}}$ (*generalized maximum-dissipation principle* (Halphen and Nguyen, 1975)). When $\mathbf{q}^d \in \partial\overline{\mathbb{E}}$, the inequality (14) indicates that $\dot{\mathbf{v}}$ belongs to the cone orthogonal to $\partial\overline{\mathbb{E}}$ at the point $\mathbf{q}^d$. When $\mathbf{q}^d \in \text{int}(\overline{\mathbb{E}})$, it forces the flow $\dot{\mathbf{v}}$ to be zero (Appendix A, item 7).

3. Prandtl–Reuss model

3.1. Perfectly plastic Prandtl–Reuss model

In order to illustrate the general procedure that is adopted hereinafter, the basic example of the Prandtl–Reuss model is considered first. The relevant state variables are the total and the plastic strain $\mathbf{v} = (\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p)$ and $\mathbf{q}^{nd} = (\boldsymbol{\sigma}^{nd}, \boldsymbol{\tau}^{nd})$ are the associated non-dissipative thermodynamic forces. The usual quadratic form of the Helmholtz free energy density Ψ is used, in order to preserve the linear dependence of all non-dissipative forces with respect to state variables:

$$\Psi = \frac{1}{2}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) : \mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) \tag{15}$$

For isotropic materials, the fourth-order tensor of the elastic moduli is equal to $\mathbf{C} = (K - \frac{2}{3}G)\mathbf{1} \otimes \mathbf{1} + 2G\mathbf{I}$, where K is the (isothermal) bulk modulus, G is the shear modulus and $\otimes$ is the direct (or outer) product of two second order tensors. The assumption of isotropy is always adopted, even if the concise symbol $\mathbf{C}$ is used. The non-dissipative forces associated with (15) can be derived by means of (5):

$$\boldsymbol{\sigma}^{nd} = \mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p), \quad \boldsymbol{\tau}^{nd} = -\mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) \tag{16}$$

The evolution of the dissipative forces $\mathbf{q}^d = (\boldsymbol{\sigma}^d, \boldsymbol{\tau}^d) \in \mathbb{S}^2 \times \mathbb{S}^2 := \mathbb{V}^*$ is defined by introducing a suitable pseudo-potential ϕ, which is a function of the fluxes $\dot{\mathbf{v}}' = (\dot{\boldsymbol{\varepsilon}}', \dot{\boldsymbol{\varepsilon}}^{p'}) \in \mathbb{S}^2 \times \mathbb{S}^2 := \mathbb{V}$ ($\times$ is the cartesian product):

$$\begin{aligned} &\phi(\dot{\boldsymbol{\varepsilon}}', \dot{\boldsymbol{\varepsilon}}^{p'}) = \sqrt{\frac{2}{3}}\sigma_y \|\dot{\boldsymbol{\varepsilon}}^{p'}\| + \mathbb{I}_{\overline{\mathbb{D}}}(\dot{\boldsymbol{\varepsilon}}', \dot{\boldsymbol{\varepsilon}}^{p'}) \\ &\overline{\mathbb{D}} = \left\{ (\dot{\boldsymbol{\varepsilon}}', \dot{\boldsymbol{\varepsilon}}^{p'}) \in \mathbb{V} \text{ such that } \operatorname{tr}(\dot{\boldsymbol{\varepsilon}}^{p'}) = 0 \right\} \end{aligned} \tag{17}$$

where $\operatorname{tr}(\mathbf{u})$ indicates the trace of the tensor $\mathbf{u} \in \mathbb{S}^2$. The norm of the second order symmetric tensor $\mathbf{u}$ is given by $\|\mathbf{u}\| = \sqrt{u_{ij}u_{ij}}$. If in addition $\operatorname{tr}(\mathbf{u}) = u_{ii} = 0$, then $\|\mathbf{u}\|^2 = 2J_2(\mathbf{u})$ where $J_2(\mathbf{u})$ is the second invariant of the deviatoric part of $\mathbf{u}$; σ_y is the one-dimensional tension stress limit and $\mathbb{I}_{\overline{\mathbb{D}}}$ is the indicator function of the set $\overline{\mathbb{D}}$, namely $\mathbb{I}_{\overline{\mathbb{D}}} = 0$ if $\operatorname{tr}(\dot{\boldsymbol{\varepsilon}}^{p'}) = 0$ and $\mathbb{I}_{\overline{\mathbb{D}}} = +\infty$ elsewhere. This set is the effective domain of ϕ (Appendix A, item 2). The pseudo-potential ϕ is a homogeneous function of order 1 with respect to $(\dot{\boldsymbol{\varepsilon}}', \dot{\boldsymbol{\varepsilon}}^{p'})$ and therefore a rate-independent constitutive behavior is expected and the dissipation Φ_m is equal to $\sqrt{\frac{2}{3}}\sigma_y \|\dot{\boldsymbol{\varepsilon}}^p\|$, where $\dot{\boldsymbol{\varepsilon}}^p$ is the actual plastic flow (Appendix A, item 6). The indicator function $\mathbb{I}_{\overline{\mathbb{D}}}$ accounts for the fact that plastic deformation occurs without volume changes (*plastic incompressibility*). This assumption is usual for metals and has been validated by experimental evidence.

The pseudo-potential ϕ^*, dual of ϕ, can be computed using the Legendre–Fenchel transform (Appendix A, item 5) and is equal to:

$$\phi^*(\boldsymbol{\sigma}^{d'}, \boldsymbol{\tau}^{d'}) = \sup_{(\dot{\boldsymbol{\varepsilon}}', \dot{\boldsymbol{\varepsilon}}^{p'}) \in \overline{\mathbb{D}}} (\boldsymbol{\sigma}^{d'} : \dot{\boldsymbol{\varepsilon}}' + \boldsymbol{\tau}^{d'} : \dot{\boldsymbol{\varepsilon}}^{p'} - \phi) = \mathbb{I}_{\overline{\mathbb{E}}}(\boldsymbol{\sigma}^{d'}, \boldsymbol{\tau}^{d'}) \tag{18}$$

The dual pseudo-potential ϕ^* is the indicator function of a closed convex set $\overline{\mathbb{E}}$. Hence, $\phi = \phi^{**}$ is the support function of the same set (Appendix A, item 6). Moreover, since ϕ does not explicitly depend on $\dot{\boldsymbol{\varepsilon}}'$, the dual pseudo-potential ϕ^* can be written as the sum of two indicator functions (Appendix A, item 7):

$$\begin{aligned}&\phi^*(\boldsymbol{\sigma}^{d'}, \boldsymbol{\tau}^{d'}) = \mathbb{I}_0(\boldsymbol{\sigma}^{d'}) + \mathbb{I}_{\mathbb{E}}(\boldsymbol{\tau}^{d'})\\ &\mathbb{E} = \left\{ \boldsymbol{\tau}^{d'} \in \mathbb{S}^2 \text{ such that } f(\boldsymbol{\tau}^{d'}) = \|\mathrm{dev}(\boldsymbol{\tau}^{d'})\| - \sqrt{\frac{2}{3}}\sigma_y \leqslant 0 \right\}\end{aligned} \tag{19}$$

where dev(**u**) is the deviatoric part of $\mathbf{u} \in \mathbb{S}^2$. The first term entails the condition $\boldsymbol{\sigma}^{d'} = \mathbf{0}$, while the other indicator function $\mathbb{I}_{\mathbb{E}}$ defines a region in the $\boldsymbol{\tau}^{d'}$ stress space. Recalling that the *actual value* of $\boldsymbol{\tau}^{d'}$, viz. $\boldsymbol{\tau}^{d}$, fulfils the condition $\boldsymbol{\tau}^d = -\boldsymbol{\tau}^{nd}$ and that the only possible value for $\boldsymbol{\sigma}^d$ is zero, using (16) it is straightforward to see that $\boldsymbol{\tau}^d = \boldsymbol{\sigma} = \boldsymbol{\sigma}^{nd}$ and that $\mathbb{E}$ can also be interpreted as a set in the $\boldsymbol{\sigma}$ stress space. The associated function *f* is known as *loading function* and the condition $f = 0$ defines the *plastic states*. The interior of $\mathbb{E}$ is associated with the *elastic states* and the whole (closed) set $\mathbb{E}$ contains all *plastically admissible states*. The actual flows $(\dot{\boldsymbol{\varepsilon}}, \dot{\boldsymbol{\varepsilon}}^p)$ can now be derived from (19) by computing the subdifferential set of ϕ^* and then considering it at $(\boldsymbol{\sigma}^{d'}, \boldsymbol{\tau}^{d'}) = (\boldsymbol{\sigma}^d, \boldsymbol{\tau}^d)$. Hence, no restrictive conditions are imposed on $\dot{\boldsymbol{\varepsilon}}$, while the plastic strain flow reads (Appendix A, item 7)

$$\dot{\boldsymbol{\varepsilon}}^p = \frac{\mathrm{dev}\,(\boldsymbol{\tau}^d)}{\|\mathrm{dev}\,(\boldsymbol{\tau}^d)\|}\dot{\lambda} = \mathbf{n}\dot{\lambda} \quad \text{with } \dot{\lambda} \geqslant 0,\; f(\boldsymbol{\tau}^d) \leqslant 0,\; \dot{\lambda} f(\boldsymbol{\tau}^d) = 0 \tag{20}$$

Observe that *f* in the loading–unloading conditions in the second row of (20) is computed at the actual stress state. The plastic multiplier $\dot{\lambda}$ is then evaluated by imposing the *consistency condition*, i.e. $\dot{\lambda}\dot{f} = 0$ (see e.g. Simo and Hughes (1988)), with

$$\dot{f} = \left[\frac{\partial f}{\partial \boldsymbol{\tau}^{d'}} : \dot{\boldsymbol{\tau}}^{d'} \right]_{\boldsymbol{\tau}^{d'} = \boldsymbol{\tau}^d} \tag{21}$$

Note that $\dot{f}$ is computed from the general expression of $f(\boldsymbol{\tau}^{d'})$ and then evaluated at the actual state $\boldsymbol{\tau}^{d'} = \boldsymbol{\tau}^d$. Consistency corresponds to the requirement that in order to have $\dot{\lambda} > 0$, the *actual* dissipative force $\boldsymbol{\tau}^d \in \partial\mathbb{E}$ cannot leave $\partial\mathbb{E}$ during the plastic flow. Hence, by using (20) and the relationship $\boldsymbol{\tau}^d = \boldsymbol{\sigma} = \mathbf{C}{:}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p)$, one has

$$\dot{f} = \frac{\mathrm{dev}\,(\boldsymbol{\tau}^d)}{\|\mathrm{dev}\,(\boldsymbol{\tau}^d)\|} : \mathrm{dev}\,(\dot{\boldsymbol{\tau}}^d) = \mathbf{n} : \mathbf{C} : \dot{\boldsymbol{\varepsilon}} - \mathbf{n} : \mathbf{C} : \mathbf{n}\,\dot{\lambda} \tag{22}$$

thus $\dot{\lambda} = H(f)\frac{\langle \mathbf{n}:\mathbf{C}:\dot{\boldsymbol{\varepsilon}}\rangle}{\mathbf{n}:\mathbf{C}:\mathbf{n}} = H(f)\langle \mathbf{n} : \dot{\boldsymbol{\varepsilon}} \rangle$, where $\langle x \rangle = \frac{x+|x|}{2}$ (*McCauley brackets*) and *H*(*f*) is the Heaviside function, equal to zero for $f < 0$ and equal to 1 elsewhere.

In summary, the Prandtl–Reuss perfectly plastic model was formulated by means of the Helmholtz free energy Ψ and the pseudo-potential ϕ; then, the dual potential ϕ^* was computed from the Legendre–Fenchel transform of ϕ; the subdifferential set of ϕ^* was used to define the fluxes and the consistency assumption led to the determination of the plastic multiplier $\dot{\lambda}$. In the next sections, this approach will be used to formulate two Prandtl–Reuss models with isotropic hardening and other more complex plasticity models, such as endochronic, NLK hardening and generalized plasticity models. In order to get this result, some non-standard expressions for the pseudo-potentials ϕ and ϕ^* are introduced.

3.2. Classical Prandtl–Reuss model with isotropic hardening

The vector of the representative state variables for a Prandtl–Reuss model with isotropic hardening is equal to $\mathbf{v} = (\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \zeta)$, while $\mathbf{q}^{nd} = (\boldsymbol{\sigma}^{nd}, \boldsymbol{\sigma}, R^{nd})$ are the associated non-dissipative forces. Moreover, $\dot{\mathbf{v}} = (\dot{\boldsymbol{\varepsilon}}, \dot{\boldsymbol{\varepsilon}}^p, \dot{\zeta}) \in \mathbb{V} = \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{R}$ is the flux vector and $\mathbf{q}^d = (\boldsymbol{\sigma}^d, \boldsymbol{\sigma}^d, R^d) \in \mathbb{V}^* = \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{R}$ contains all dissipative thermodynamic forces. The scalar internal variable ζ and the associated forces R^d and R^{nd} are introduced in order to represent the isotropic hardening. The Helmholtz free energy is assumed to be of the form

$$\Psi = \frac{1}{2}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) : \mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) + \xi(\zeta) \tag{23}$$

where $\xi(\zeta)$ is a scalar function such that $\xi(0) = 0$ and $\frac{d\xi}{d\zeta}(0) = 0$. It follows that

$$\boldsymbol{\sigma}^{nd} = \mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p), \quad \boldsymbol{\tau}^{nd} = -\mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p), \quad R^{nd} = \frac{d\xi}{d\zeta}(\zeta) \tag{24}$$

The pseudo-potential is assumed of the following form:

$$\begin{aligned} &\phi(\dot{\boldsymbol{\varepsilon}}', \dot{\boldsymbol{\varepsilon}}^{p'}, \dot{\zeta}') = \sqrt{\frac{2}{3}}\sigma_y \dot{\zeta}' + \mathbb{I}_{\overline{\mathbb{D}}}(\dot{\boldsymbol{\varepsilon}}', \dot{\boldsymbol{\varepsilon}}^{p'}, \dot{\zeta}') \\ &\overline{\mathbb{D}} = \left\{ (\dot{\boldsymbol{\varepsilon}}', \dot{\boldsymbol{\varepsilon}}^{p'}, \dot{\zeta}') \in \mathbb{V} \text{ such that } \operatorname{tr}(\dot{\boldsymbol{\varepsilon}}^{p'}) = 0 \text{ and } \dot{\zeta}' \geqslant \|\dot{\boldsymbol{\varepsilon}}^{p'}\| \right\} \end{aligned} \tag{25}$$

The first term in the expression of ϕ is the same as in the perfectly-plastic model when $\dot{\zeta}'$ is equal to the norm of $\dot{\boldsymbol{\varepsilon}}^{p'}$. The second term, that is the indicator function $\mathbb{I}_{\overline{\mathbb{D}}}$, depends not only on $\operatorname{tr}(\dot{\boldsymbol{\varepsilon}}^{p'})$, but also on the flow $\dot{\zeta}'$, which is forced be greater or equal than the norm of the plastic strain flow. This inequality guarantees that $\dot{\zeta}'$ and ϕ are non-negative and entails that $\overline{\mathbb{D}}$ is convex and closed (see Appendix A, item 1 and Fig. 1a, which illustrates the projection of $\overline{\mathbb{D}}$ on the $(\dot{\boldsymbol{\varepsilon}}^{p'}, \dot{\zeta}')$-plane for the tension-compression case). The dual pseudo-potential is different from (18), due to the presence of the dissipative force $R^{d'}$ associated with $\dot{\zeta}'$:

$$\phi^*(\boldsymbol{\sigma}^{d'}, \boldsymbol{\tau}^{d'}, R^{d'}) = \sup_{(\dot{\boldsymbol{\varepsilon}}', \dot{\boldsymbol{\varepsilon}}^{p'}, \dot{\zeta}') \in \overline{\mathbb{D}}} (\boldsymbol{\sigma}^{d'} : \dot{\boldsymbol{\varepsilon}}' + \boldsymbol{\tau}^{d'} : \dot{\boldsymbol{\varepsilon}}^{p'} + R^{d'}\dot{\zeta}' - \phi) = \mathbb{I}_0(\boldsymbol{\sigma}^{d'}) + \mathbb{I}_{\mathbb{E}}(\boldsymbol{\tau}^{d'}, R^{d'}) \tag{26}$$

where $\mathbb{E} = \{(\boldsymbol{\tau}^{d'}, R^{d'}) \in \mathbb{S}^2 \times \mathbb{R} \text{ such that } f(\boldsymbol{\tau}^{d'}, R^{d'}) \leqslant 0\}$ and

$$f(\boldsymbol{\tau}^{d'}, R^{d'}) = \|\operatorname{dev}(\boldsymbol{\tau}^{d'})\| - \left(\sqrt{\frac{2}{3}}\sigma_y - R^{d'}\right) \tag{27}$$

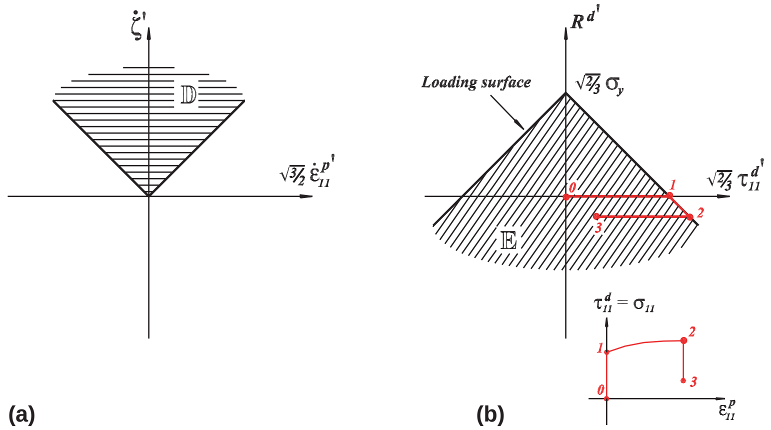

Fig. 1. Classical Prandtl–Reuss model. Tension-compression case. (a) Projection of the pseudo-potential effective domain $\overline{\mathbb{D}}$ on the $(\dot{\boldsymbol{\varepsilon}}^{p'}, \dot{\zeta}')$-plane. This set is indicated by $\mathbb{D}$. (b) Domain $\mathbb{E}$ associated with the dual pseudo-potential ϕ^*.

The loading function f defines a convex and closed region $\mathbb{E}$ in the $(\boldsymbol{\tau}^{d'}, R^{d'})$ space, where the *actual value* of $R^{d'}$, viz. $R^d = -R^{nd} = -\frac{\mathrm{d}\xi(\zeta)}{\mathrm{d}\zeta}$ governs isotropic hardening (or softening). The limit stress becomes greater than its initial value $\sqrt{\frac{2}{3}}\sigma_y$ when $\frac{\mathrm{d}\xi(\zeta)}{\mathrm{d}\zeta} \geqslant 0$ and less when $\frac{\mathrm{d}\xi(\zeta)}{\mathrm{d}\zeta} \leqslant 0$. Fig. 1b illustrates the set $\mathbb{E}$. The flow rules follow from the generalized normality conditions:

$$\dot{\boldsymbol{\varepsilon}}^p = \frac{\operatorname{dev}(\boldsymbol{\tau}^d)}{\|\operatorname{dev}(\boldsymbol{\tau}^d)\|}\dot{\lambda} = \mathbf{n}\dot{\lambda}, \quad \dot{\zeta} = \dot{\lambda} \quad \text{with } \dot{\lambda} \geqslant 0,\ f \leqslant 0,\ \dot{\lambda}f = 0 \tag{28}$$

The flow of the internal variable ζ is equal to the plastic multiplier $\dot{\lambda}$, which can be evaluated by imposing the consistency condition:

$$\dot{f} = \left[\frac{\partial f}{\partial \boldsymbol{\tau}^{d'}} : \dot{\boldsymbol{\tau}}^{d'} + \frac{\partial f}{\partial R^{d'}}\dot{R}^{d'}\right]_{(\boldsymbol{\tau}^{d'}=\boldsymbol{\tau}^d, R^{d'}=R^d)} = 0 \tag{29}$$

It follows that

$$\dot{\lambda} = H(f)\frac{\langle \mathbf{n} : \mathbf{C} : \dot{\boldsymbol{\varepsilon}}\rangle}{\mathbf{n} : \mathbf{C} : \mathbf{n} + \frac{\mathrm{d}^2\xi(\zeta)}{\mathrm{d}\zeta^2}} = H(f)\frac{\langle \mathbf{n} : \dot{\boldsymbol{\varepsilon}}\rangle}{1 + \frac{1}{2G}\frac{\mathrm{d}^2\xi(\zeta)}{\mathrm{d}\zeta^2}} \tag{30}$$

where $H(f)$ and $\langle\ \rangle$ still indicate the Heaviside function and McCauley brackets, respectively.

3.3. Modified Prandtl–Reuss model with isotropic hardening

The classical model of the previous section can be extended as follows. Assume the state variables $\mathbf{v} = (\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \zeta)$ and let the Helmholtz energy be equal to

$$\Psi = \frac{1}{2}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) : \mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) + \xi(\zeta) \tag{31}$$

As a result, the non-dissipative forces are the same as in Eq. (24). Then, a generalized definition of the pseudo-potential ϕ is adopted:

$$\begin{aligned} &\phi(\dot{\boldsymbol{\varepsilon}}', \dot{\boldsymbol{\varepsilon}}^{p'}, \dot{\zeta}'; \zeta) = \left(\sqrt{\frac{2}{3}}\sigma_y g(\zeta) - \frac{\mathrm{d}\xi(\zeta)}{\mathrm{d}\zeta}\right)\dot{\zeta}' + \mathbb{I}_{\overline{\mathbb{D}}}(\dot{\boldsymbol{\varepsilon}}', \dot{\boldsymbol{\varepsilon}}^{p'}, \dot{\zeta}') \\ &\overline{\mathbb{D}} = \{ (\dot{\boldsymbol{\varepsilon}}', \dot{\boldsymbol{\varepsilon}}^{p'}, \dot{\zeta}') \in \mathbb{V} \text{ such that } \operatorname{tr}(\dot{\boldsymbol{\varepsilon}}^{p'}) = 0 \text{ and } \dot{\zeta}' \geqslant \|\dot{\boldsymbol{\varepsilon}}^{p'}\| \} \end{aligned} \tag{32}$$

In this case, ϕ explicitly depends on the internal variable ζ, by means of $\frac{\mathrm{d}\xi(\zeta)}{\mathrm{d}\zeta}$ and of the function $g(\zeta)$, positive and such that $g(0) = 1$. In the particular case where $g(\zeta) = 1 + \frac{\mathrm{d}\xi(\zeta)}{\mathrm{d}\zeta}\sqrt{\frac{3}{2}}\frac{1}{\sigma_y}$, the classical expression given in Eq. (25) is recovered. The dual pseudo-potential ϕ^* can be evaluated from the standard procedure, thus yielding:

$$\phi^*(\boldsymbol{\sigma}^{d'}, \boldsymbol{\tau}^{d'}, R^{d'}; \zeta) = \mathbb{I}_0(\boldsymbol{\sigma}^{d'}) + \mathbb{I}_{\mathbb{E}}(\boldsymbol{\tau}^{d'}, R^{d'}; \zeta) \tag{33}$$

where $\mathbb{E} = \{(\boldsymbol{\tau}^{d'}, R^{d'}) \in \mathbb{S}^2 \times \mathbb{R} \text{ such that } f(\boldsymbol{\tau}^{d'}, R^{d'}; \zeta) \leqslant 0\}$ and

$$f(\boldsymbol{\tau}^{d'}, R^{d'}; \zeta) = \|\operatorname{dev}(\boldsymbol{\tau}^{d'})\| - \left(\sqrt{\frac{2}{3}}\sigma_y g(\zeta) - \frac{\mathrm{d}\xi(\zeta)}{\mathrm{d}\zeta} - R^{d'}\right) \tag{34}$$

In Fig. 2, the projection of $\overline{\mathbb{D}}$ on the $(\dot{\boldsymbol{\varepsilon}}^{p'}, \dot{\zeta}')$-plane and the set $\mathbb{E}$ are depicted for the tension-compression case, with the assumption $\xi(\zeta) = 0$. The flow rules are the same as in the previous case and they are reported below for completeness:

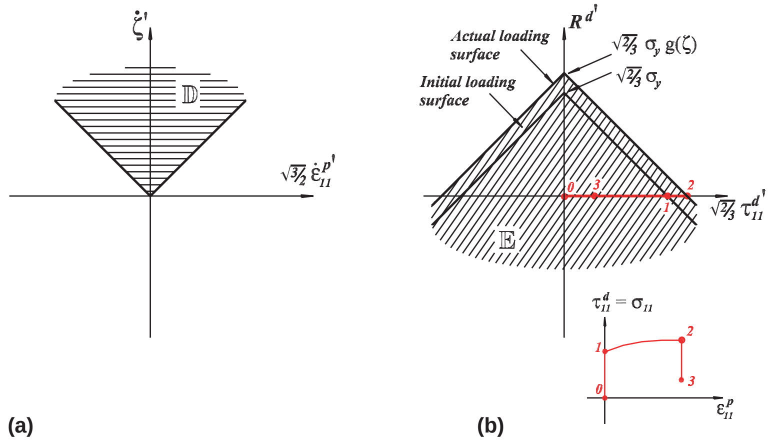

Fig. 2. Modified Prandtl–Reuss model. Tension-compression case with $\xi(\zeta) = 0$. (a) Projection of the pseudo-potential effective domain $\overline{\mathbb{D}}$ on the $(\dot{\varepsilon}^{p'}, \dot{\zeta}')$-plane. This set is indicated by $\mathbb{D}$. (b) Different configurations of the domain $\mathbb{E}$. The position of $\mathbb{E}$ changes according to the value of the internal variable ζ. The point $(\boldsymbol{\tau}^d, R^d)$, representing the actual state, always lies on the axis $R^{d'} = 0$.

$$\dot{\boldsymbol{\varepsilon}}^p = \frac{\mathrm{dev}\,(\boldsymbol{\tau}^d)}{\|\mathrm{dev}\,(\boldsymbol{\tau}^d)\|}\dot{\lambda} = \mathbf{n}\dot{\lambda}, \quad \dot{\zeta} = \dot{\lambda} \quad \text{with } \dot{\lambda} \geqslant 0, \quad f \leqslant 0, \ \dot{\lambda} f = 0 \tag{35}$$

In this case, $\dot{f}$ has to be computed accounting for the state variables. Hence, consistency condition reads

$$\dot{f} = \left[\frac{\partial f}{\partial \boldsymbol{\tau}^{d'}} : \dot{\boldsymbol{\tau}}^{d'} + \frac{\partial f}{\partial R^{d'}}\dot{R}^{d'} + \frac{\partial f}{\partial \zeta}\dot{\zeta}\right]_{(\boldsymbol{\tau}^{d'} = \boldsymbol{\tau}^d, R^{d'} = R^d)} = 0 \tag{36}$$

and the plastic multiplier becomes equal to:

$$\dot{\lambda} = H(f)\frac{\langle \mathbf{n} : \mathbf{C} : \dot{\boldsymbol{\varepsilon}}\rangle}{\mathbf{n} : \mathbf{C} : \mathbf{n} + \sqrt{\frac{2}{3}}\sigma_y \frac{\mathrm{d}g(\zeta)}{\mathrm{d}\zeta}} = H(f)\frac{\langle \mathbf{n} : \dot{\boldsymbol{\varepsilon}}\rangle}{1 + \sqrt{\frac{2}{3}}\frac{\sigma_y}{2G}\frac{\mathrm{d}g(\zeta)}{\mathrm{d}\zeta}} \tag{37}$$

provided that $1 + \sqrt{\frac{2}{3}}\frac{\sigma_y}{2G}\frac{\mathrm{d}g(\zeta)}{\mathrm{d}\zeta} > 0$. This condition does not prevent softening, which occurs when $\frac{\mathrm{d}g(\zeta)}{\mathrm{d}\zeta} \leqslant 0$.

The comparison of Eqs. (27) and (34) proves to be very interesting. First, the usual loading function only depends on the dissipative forces, while f in Eq. (34) is also related to the internal variable ζ. Moreover, since $R^d = -R^{nd} = -\frac{\mathrm{d}\xi(\zeta)}{\mathrm{d}\zeta}$, the loading function (34) at $(\boldsymbol{\tau}^d, R^d)$ becomes

$$f(\boldsymbol{\tau}^d, R^d; \zeta) = \|\mathrm{dev}\,(\boldsymbol{\tau}^d)\| - \sqrt{\frac{2}{3}}\sigma_y g(\zeta) \tag{38}$$

This expression shows that the actual limit stress is equal to $\sqrt{\frac{2}{3}}\sigma_y g(\zeta)$ and is independent from the function $\xi(\zeta)$ introduced in the Helmholtz energy density (this is not the case for the classical Prandtl–Reuss model).

The difference between the two Prandtl–Reuss models can be also explained in terms of mechanical dissipation Φ_m. For the modified Prandtl–Reuss model, it is equal to

$$\Phi_m = \left(\sqrt{\frac{2}{3}}\sigma_y g(\zeta) - \frac{\mathrm{d}\xi(\zeta)}{\mathrm{d}\zeta}\right)\dot{\zeta} \tag{39}$$

which is non-negative provided that $\frac{\mathrm{d}\xi(\zeta)}{\mathrm{d}\zeta} \leqslant \sqrt{\frac{2}{3}}\sigma_y g(\zeta)$. The case of a mono-dimensional monotonic loading is depicted in Fig. 3. The standard Prandtl–Reuss model is characterized by the fact that the energy $R^d\dot{\zeta}$

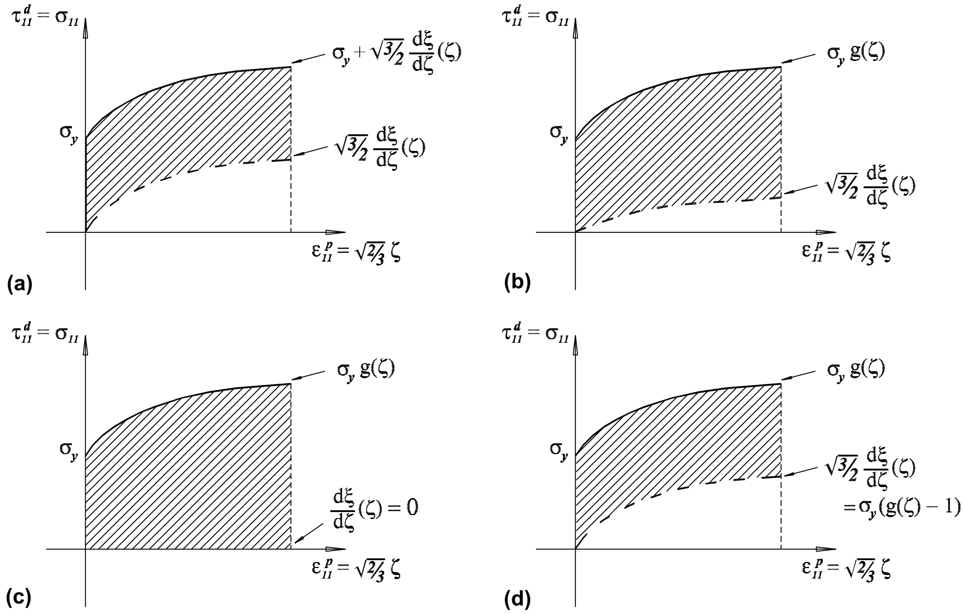

Fig. 3. Mechanical dissipation for the case of simple tension. The hatched area is the energy $\int \Phi_m(t)\,dt$ dissipated during the monotonic loading. (a) Classical Prandtl–Reuss model. (b) Modified Prandtl–Reuss model. (c) Modified Prandtl–Reuss model with $\xi(\zeta) = \frac{d\xi}{d\zeta}(\zeta) = 0$. (d) Modified Prandtl–Reuss model with $\sqrt{\frac{3}{2}}\frac{d\xi}{d\zeta}(\zeta) = \sigma_y(g(\zeta) - 1)$: the classical model is recovered.

associated to isotropic hardening is not dissipated. For this reason it is sometimes referred as *energy blocked in dislocations* (Lemaitre and Chaboche, 1990, p. 402). Hence, the mechanical dissipation is equal to $\sqrt{\frac{2}{3}}\sigma_y\dot{\zeta}$ for any function $\xi(\zeta)$. Conversely, for the modified Prandtl–Reuss model the amount of mechanical dissipation depends, for a given function $g(\zeta)$, on the choice of $\xi(\zeta)$. Fig. 3b reports the case of generic functions $g(\zeta)$ and $\xi(\zeta)$. Fig. 3c and d correspond to $\xi(\zeta) = 0$ and to the case where the modified model is equal to the classical one, respectively.

3.4. Multi-layer models of Prandtl–Reuss type

Modified Prandtl–Reuss models, defined by Eqs. (31) and (32), can be directly extended to multi-layer models (Besseling, 1958). They consist of a system of N elastoplastic elements connected in parallel. When every individual elements are Prandtl–Reuss models, the corresponding multi-layer model is indicated as of the *Prandtl–Reuss type*. This is the case in the present section. Hence, let

$$\Psi = \sum_{i=1}^{N} \Psi_i = \sum_{i=1}^{N} \left[\frac{1}{2}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_i^p) : \mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_i^p) + \xi_i(\zeta_i)\right] \tag{40}$$

be the Helmholtz energy density, defined as the sum of N expressions of the type (31). The internal variable $\boldsymbol{\varepsilon}_i^p$ is the plastic strain of the generic element i, while ζ_i is the scalar variable associated with the isotropic hardening of the same element. All elements have by definition the same elastic modulus tensor, chosen to be equal to $\mathbf{C} = \frac{1}{N}[(K - \frac{2}{3}G)\mathbf{1} \otimes \mathbf{1} + 2G\mathbf{I}]$. The non-dissipative thermodynamic forces read:

$$\boldsymbol{\sigma}^{nd} = \sum_{i=1}^{N} \mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_i^p), \quad \boldsymbol{\tau}_i^{nd} = -\mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_i^p), \quad R_i^{nd} = \frac{d\xi_i(\zeta_i)}{d\zeta_i} \tag{41}$$

Let us introduce the pseudo-potential ϕ as the sum of N *independent* functions of the type (32):

$$\phi = \sum_{i=1}^{N} \phi_i(\dot{\boldsymbol{\varepsilon}}', \dot{\boldsymbol{\varepsilon}}_i^{p'}, \dot{\zeta}_i'; \zeta_i) = \sum_{i=1}^{N} \left[\left(\sqrt{\frac{2}{3}} \sigma_{yi} g_i(\zeta_i) - \frac{\mathrm{d}\xi_i(\zeta_i)}{\mathrm{d}\zeta_i} \right) \dot{\zeta}_i' + \mathbb{I}_{\overline{\mathbb{D}}_i}(\dot{\boldsymbol{\varepsilon}}', \dot{\boldsymbol{\varepsilon}}_i^{p'}, \dot{\zeta}_i') \right]$$
$$\overline{\mathbb{D}}_i = \{ (\dot{\boldsymbol{\varepsilon}}', \dot{\boldsymbol{\varepsilon}}_i^{p'}, \dot{\zeta}_i') \in \mathbb{V} \quad \text{such that } \operatorname{tr}(\dot{\boldsymbol{\varepsilon}}_i^{p'}) = 0 \text{ and } \dot{\zeta}_i' \geqslant \|\dot{\boldsymbol{\varepsilon}}_i^{p'}\| \} \tag{42}$$

The limit stresses σ_{yi} as well as the isotropic hardening functions $g_i(\zeta_i)$ are, in general, distinct. The conjugated pseudo-potential is in turn the sum of N independent functions, i.e. $\phi^* = \sum_{i=1}^{N} \phi_i^*$ with

$$\phi_i^*(\boldsymbol{\sigma}^{d'}, \boldsymbol{\tau}_i^{d'}, R_i^{d'}) = \sup_{(\dot{\boldsymbol{\varepsilon}}', \dot{\boldsymbol{\varepsilon}}_i^{p'}, \dot{\zeta}_i') \in \overline{\mathbb{D}}_i} (\boldsymbol{\sigma}^{d'} : \dot{\boldsymbol{\varepsilon}}' + \boldsymbol{\tau}_i^{d'} : \dot{\boldsymbol{\varepsilon}}_i^{p'} + R_i^{d'} \dot{\zeta}_i' - \phi_i) = \mathbb{I}_0(\boldsymbol{\sigma}^{d'}) + \mathbb{I}_{\mathbb{E}_i}(\boldsymbol{\tau}_i^{d'}, R_i^{d'}) \tag{43}$$

where $\mathbb{E}_i = \{(\boldsymbol{\tau}_i^{d'}, R_i^{d'}) \in \mathbb{S}^2 \times \mathbb{R}$ such that $f_i(\boldsymbol{\tau}_i^{d'}, R_i^{d'}; \zeta_i) \leqslant 0\}$ and

$$f_i(\boldsymbol{\tau}_i^{d'}, R_i^{d'}; \zeta_i) = \|\operatorname{dev}(\boldsymbol{\tau}_i^{d'})\| - \sqrt{\frac{2}{3}} \sigma_{yi} g_i(\zeta_i) + R_i^{d'} + \frac{\mathrm{d}\xi_i(\zeta_i)}{\mathrm{d}\zeta_i} \tag{44}$$

Therefore, N independent loading surfaces have been defined. Using the standard procedure based on the normality assumption, N pairs of flow rules of the type (35) can be derived:

$$\dot{\boldsymbol{\varepsilon}}_i^p = \frac{\operatorname{dev}(\boldsymbol{\tau}_i^d)}{\|\operatorname{dev}(\boldsymbol{\tau}_i^d)\|} \dot{\lambda}_i = \mathbf{n}_i \dot{\lambda}_i, \quad \dot{\zeta}_i = \dot{\lambda}_i \quad \text{with } \dot{\lambda}_i \geqslant 0,\ f_i \leqslant 0,\ \dot{\lambda}_i f_i = 0 \tag{45}$$

Moreover, by imposing the consistency conditions and accounting for Eqs. (41) as well as the identities $\boldsymbol{\tau}_i^d = -\boldsymbol{\tau}_i^{nd}$ and $R_i^d = -R_i^{nd}$, each plastic multiplier can be easily determined by an expression of the type (37):

$$\dot{\lambda}_i = H(f_i) \frac{\langle \mathbf{n}_i : \mathbf{C} : \dot{\boldsymbol{\varepsilon}} \rangle}{\mathbf{n}_i : \mathbf{C} : \mathbf{n}_i + \sqrt{\frac{2}{3}} \sigma_{yi} \frac{\mathrm{d}g_i(\zeta_i)}{\mathrm{d}\zeta_i}} = H(f_i) \frac{\langle \mathbf{n}_i : \dot{\boldsymbol{\varepsilon}} \rangle}{1 + \sqrt{\frac{2}{3}} \frac{\sigma_{yi}}{2G} \frac{\mathrm{d}g_i(\zeta_i)}{\mathrm{d}\zeta_i}} \tag{46}$$

provided that $1 + \sqrt{\frac{2}{3}} \frac{\sigma_{yi}}{2G} \frac{\mathrm{d}g_i(\zeta_i)}{\mathrm{d}\zeta_i} > 0$.

The Distributed Element Model (Iwan, 1966; Chiang and Beck, 1994) is recovered when $g_i(\zeta_i) = 1$ and $\xi_i(\zeta_i) = 0$.

4. Endochronic theory

Endochronic theory was first formulated by Valanis (1971), who suggested the use of a positive scalar variable ϑ, called *intrinsic time*, in the definition of constitutive laws of plasticity models. The evolution laws are described by convolution integrals involving past values of the state variable $\boldsymbol{\varepsilon}$ and a suitable scalar functions depending on ϑ called *memory kernel*. When the memory kernel is exponential, the integral expressions can be rewritten as simple differential equations, which, for an initially isotropic endochronic material fulfilling the plastic incompressibility assumption, read:

$$\begin{cases} \operatorname{tr}(\dot{\boldsymbol{\sigma}}) = 3K \operatorname{tr}(\dot{\boldsymbol{\varepsilon}}) \\ \operatorname{dev}(\dot{\boldsymbol{\sigma}}) = 2G \operatorname{dev}(\dot{\boldsymbol{\varepsilon}}) - \beta \operatorname{dev}(\boldsymbol{\sigma}) \dot{\vartheta} \end{cases} \tag{47}$$

with $\beta > 0$. These relationships are equivalent to:

$$\begin{cases} \boldsymbol{\sigma} = \mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p), \\ \mathbf{C} = (K - \frac{2}{3} G) \mathbf{1} \otimes \mathbf{1} + 2G\mathbf{I}, \\ \operatorname{tr}(\dot{\boldsymbol{\varepsilon}}^p) = 0 \text{ and } \dot{\boldsymbol{\varepsilon}}^p = \dfrac{\operatorname{dev}(\boldsymbol{\sigma})}{2G/\beta} \dot{\vartheta} \end{cases} \tag{48}$$

where $\dot{\vartheta} \geqslant 0$ is the time-derivative of the intrinsic time. The simplest choice for the intrinsic time flow is $\dot{\vartheta} = \|\mathrm{dev}\,(\dot{\boldsymbol{\varepsilon}})\|$ (Valanis, 1971). However, more complex definitions can be given, such as:

$$\dot{\vartheta} = \frac{\dot{\zeta}}{g(\zeta)} = f_1(\zeta)\dot{\zeta} \quad \text{with } \dot{\zeta} = \|\mathrm{dev}\,(\dot{\boldsymbol{\varepsilon}})\| \tag{49}$$

where ζ is the *intrinsic time scale* and the positive function $f_1(\zeta) = 1/g(\zeta)$, such that $f_1(0) = 1$, is sometimes called hardening-softening function (Bažant and Bath, 1976).

4.1. A new formulation of endochronic models

In this section, the endochronic model defined by Eqs. (48) is innovatively described by its Helmholtz free energy and a suitable pseudo-potential associated with generalized normality conditions. This approach allows for insightful comparisons between endochronic models and Prandtl–Reuss models. The main implications will be discussed later. Let $\mathbf{v} = (\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \zeta)$ and $\mathbf{q}^{nd} = (\boldsymbol{\sigma}^{nd}, \boldsymbol{\tau}^{nd}, R^{nd})$ be the assumed state variables and the associated non-dissipative thermodynamic forces, respectively. They are the same as in the Prandtl–Reuss model with isotropic hardening. The Helmholtz free energy Ψ reads:

$$\Psi = \frac{1}{2}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) : \mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) \tag{50}$$

This form is a particular case of the one originally proposed by Valanis (1971), since only one tensorial internal variable, the plastic strain, is considered here. The first two non-dissipative forces $\boldsymbol{\sigma}^{nd}$ and $\boldsymbol{\tau}^{nd}$ are the same as in Eq. (24), while $R^{nd} = 0$ since Ψ is assumed to be independent of the scalar variable ζ. The pseudo-potential is defined as follows:

$$\begin{aligned}
&\phi(\dot{\boldsymbol{\varepsilon}}', \dot{\boldsymbol{\varepsilon}}^{p\prime}, \dot{\zeta}'; \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \zeta) = \frac{\|\mathrm{dev}\,[\mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p)]\|^2}{2Gg(\zeta)/\beta}\dot{\zeta}' + \mathbb{I}_{\overline{\mathbb{D}}}(\dot{\boldsymbol{\varepsilon}}', \dot{\boldsymbol{\varepsilon}}^{p\prime}, \dot{\zeta}'; \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \zeta) \\
&\overline{\mathbb{D}} = \left\{ \begin{array}{ll} (\dot{\boldsymbol{\varepsilon}}', \dot{\boldsymbol{\varepsilon}}^{p\prime}, \dot{\zeta}') \in \mathbb{V} & \text{such that} \\ \mathrm{tr}\,(\dot{\boldsymbol{\varepsilon}}^{p\prime}) = 0, & \dot{\boldsymbol{\varepsilon}}^{p\prime} = \dfrac{\mathrm{dev}\,[\mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p)]}{\frac{2G}{\beta}g(\zeta)}\dot{\zeta}', \quad \dot{\zeta}' \geqslant 0 \end{array} \right\}
\end{aligned} \tag{51}$$

The first term of ϕ, in which the stress $\boldsymbol{\sigma}^{nd} = \mathbf{C}:(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p)$ is written as a function of the state variables, is equal to the intrinsic dissipation Φ_m when $\dot{\zeta}'$ assumes the actual value $\dot{\zeta}$. The first condition associated with the closed convex set $\overline{\mathbb{D}}$ introduces the plastic incompressibility assumption, while the second condition characterizes the plastic strain flow of endochronic theory, as it can be seen by comparing it to Eqs. (48) and (49). Finally, the positivity of $\dot{\zeta}'$ is imposed in order to guarantee that ϕ is positive. Using the language of the endochronic theory, the internal variable ζ corresponds to the intrinsic time scale, while the intrinsic time ϑ is defined by its flow $\dot{\vartheta} = \dot{\zeta}/g(\zeta)$. The variable ζ does not directly appear in the Helmholtz free energy density and its associated thermodynamic forces, dissipative and non-dissipative, are thus zero. However, ζ is not zero during the plastic evolution and plays an important role in the definition of $\dot{\boldsymbol{\varepsilon}}^p$.

The conjugated pseudo-potential is, in this case, of the following form:

$$\phi^*(\boldsymbol{\sigma}^{d\prime}, \boldsymbol{\tau}^{d\prime}, R^{d\prime}; \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \zeta) = \sup_{(\dot{\boldsymbol{\varepsilon}}', \dot{\boldsymbol{\varepsilon}}^{p\prime}, \dot{\zeta}') \in \overline{\mathbb{D}}} (\boldsymbol{\sigma}^{d\prime} : \dot{\boldsymbol{\varepsilon}}' + \boldsymbol{\tau}^{d\prime} : \dot{\boldsymbol{\varepsilon}}^{p\prime} + R^{d\prime}\dot{\zeta}' - \phi) = \mathbb{I}_0(\boldsymbol{\sigma}^{d\prime}) + \mathbb{I}_{\mathbb{E}}(\boldsymbol{\tau}^{d\prime}, R^{d\prime}; \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \zeta) \tag{52}$$

where $\mathbb{E} = \{(\boldsymbol{\tau}^{d\prime}, R^{d\prime}) \in \mathbb{S}^2 \times \mathbb{R}$ such that $f(\boldsymbol{\tau}^{d\prime}, R^{d\prime}; \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \zeta) \leqslant 0\}$ and

$$f(\boldsymbol{\tau}^{d\prime}, R^{d\prime}; \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \zeta) = \frac{\mathrm{dev}\,(\boldsymbol{\tau}^{d\prime}) : \mathrm{dev}\,[\mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p)]}{2Gg(\zeta)/\beta} - \frac{\|\mathrm{dev}\,[\mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p)]\|^2}{2Gg(\zeta)/\beta} + R^{d\prime} \tag{53}$$

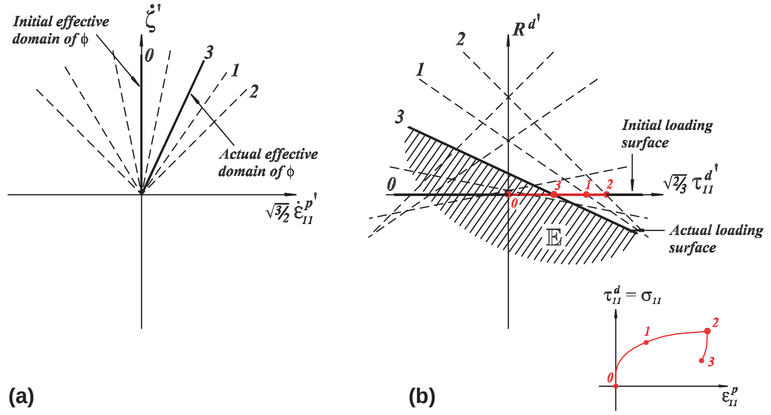

Fig. 4. Endochronic model. Tension-compression case with $g(\zeta) = 1$. (a) Several configurations of the set $\mathbb{D}$, which is the projection of the pseudo-potential effective domain $\overline{\mathbb{D}}$ on the $(\dot{\boldsymbol{\varepsilon}}^{p'}, \dot{\zeta}')$-plane. (b) Configurations of the convex set $\mathbb{E}$ associated with those of $\mathbb{D}$. The point $(\boldsymbol{\tau}^d, R^d)$, representing the actual state, always lies on the axis $R^{d'} = 0$.

The expression (53) defines the *loading function of endochronic models*. It is associated with a set $\mathbb{E}$ in the $(\boldsymbol{\tau}^{d'}, R^{d'})$ space. In Fig. 4 this set is represented in the case of tension-compression with $g(\zeta) = 1$, together with the projection of $\overline{\mathbb{D}}$ on the $(\dot{\boldsymbol{\varepsilon}}^{p'}, \dot{\zeta}')$-plane. This last set is indicated by $\mathbb{D}$. Some important remarks have to be made. First, as the system evolves, both sets change, due to their dependence on the internal variables. At every instantaneous configurations, the set $\mathbb{D}$ is a straight line starting from the origin. The corresponding sets $\mathbb{E}$ are half-planes orthogonal to $\mathbb{D}$. Moreover, Eq. (50) entails that $R^{nd} = -R^d = 0$ and, accounting for the indicator function $\mathbb{I}_0(\boldsymbol{\sigma}^{d'})$ in (52), it also leads to

$$\boldsymbol{\tau}^d = -\boldsymbol{\tau}^{nd} = \boldsymbol{\sigma} = \mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) \tag{54}$$

Therefore, at the actual stress state $(\boldsymbol{\tau}^d, R^d)$ the loading function f *is always equal to zero*. In other words, $(\boldsymbol{\tau}^d, R^d)$ always belongs to $\partial\mathbb{E}$, during both loading and unloading phases, and *all the states are plastic states*. The normality conditions lead to the endochronic flow rules:

$$\dot{\boldsymbol{\varepsilon}}^p = \frac{\mathrm{dev}\,[\mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p)]}{2Gg(\zeta)/\beta}\dot{\lambda}, \quad \dot{\zeta} = \dot{\lambda} \quad \text{with } \dot{\lambda} \geqslant 0 \tag{55}$$

Eqs. (52), (53) and (55) prove that endochronic models are associative in generalized sense. Moreover, since f is always equal to zero at the actual state, the loading–unloading conditions reduce to the requirement of the plastic multiplier $\dot{\lambda}$ to be non-negative (see the inequality in (55)). In addition, the time derivative $\dot{f}$ at $(\boldsymbol{\tau}^d, R^d)$, computed accounting for the fact that f also depends on $\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon}^p$ and ζ, is also equal to zero and therefore, *the consistency condition is automatically fulfilled and cannot be used to compute* $\dot{\lambda}$.

This situation is typical of endochronic theory and entails that the plastic multiplier $\dot{\lambda} = \dot{\zeta}$ has to be defined by an additional assumption. When the function $g(\zeta)$ is also fixed, the plastic flow $\dot{\boldsymbol{\varepsilon}}^p$ and the intrinsic time flow $\dot{\vartheta} = \frac{\dot{\zeta}}{g(\zeta)}$ are then known. The standard choices are $g(\zeta) = 1$ and $\dot{\vartheta} = \dot{\zeta} = \|\mathrm{dev}\,(\dot{\boldsymbol{\varepsilon}})\|$. It has been shown by Erlicher and Point (2004) that more complex definitions can be chosen, such as $g(\zeta) = 1$ and

$$\dot{\vartheta} = \dot{\zeta} = \|\mathrm{dev}\,(\boldsymbol{\tau}^d)\|^{n-2}\left(1 + \frac{\gamma}{\beta}\,\mathrm{sign}\,(\mathrm{dev}\,(\boldsymbol{\tau}^d) \colon \dot{\boldsymbol{\varepsilon}})\right)|\mathrm{dev}\,(\boldsymbol{\tau}^d) \colon \dot{\boldsymbol{\varepsilon}}|, \quad -\beta \leqslant \gamma \leqslant \beta,\ n > 0 \tag{56}$$

which effectively lead to the Karray–Bouc–Casciati model (Karray and Bouc, 1989; Casciati, 1989). It must be noticed that both flows $\dot{\boldsymbol{\varepsilon}}^p$ and $\dot{\zeta}$ can be different from zero during unloading phases, i.e. when $\mathrm{dev}\,(\boldsymbol{\tau}^d) \colon \dot{\boldsymbol{\varepsilon}} < 0$. This situation, which is not possible in classical plasticity, occurs when $\gamma \neq \beta$. Fig. 5a

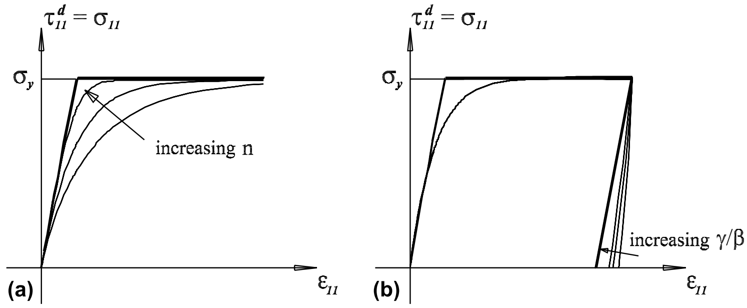

Fig. 5. Endochronic Karray–Bouc–Casciati model (thin lines) vs. Prandtl–Reuss model (thick line). Tension-compression case with $g(\zeta) = 1$ and $\xi(\zeta) = 0$. (a) Influence of the parameter n on loading branches. (b) Influence of the γ/β ratio on unloading branches. The slope at $\sigma_{11} = 0$ is the same for all γ/β values.

illustrates for the mono-dimensional case the effect of n for given values of the other parameters: in the limit of increasing n-values the Prandtl–Reuss model is retrieved. Fig. 5b shows unloading branches for different γ/β ratios, the other parameters being fixed: plastic strains may occur and tend to zero when γ/β tends to 1.

4.2. Endochronic theory vs. Prandtl–Reuss model

Consider the endochronic model, as formulated in the previous section, and the *modified* Prandtl–Reuss model. The significant state variables $\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon}^p$ and ζ are the same in both cases. Moreover, Eqs. (31) and (50) show that the Helmholtz free energies differ only by the term $\xi(\zeta)$, which is zero in endochronic theory. The main differences concern pseudo-potentials, as seen comparing Eqs. (32) and (51). However, the strict relationship between the two models can be highlighted by imposing that $\dot{\zeta}' = \|\dot{\boldsymbol{\varepsilon}}^{p'}\|$ in (51): when $\dot{\zeta}' > 0$, the condition $\|\mathrm{dev}(\mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p))\| = \frac{2G}{\beta} g(\zeta)$ must be fulfilled, while for $\dot{\zeta}' = 0$ there is no limitation on $\mathrm{dev}(\mathbf{C}:(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p))$. As a result, the endochronic pseudo-potential (51) becomes equal to

$$\begin{aligned}&\tilde{\phi}(\dot{\boldsymbol{\varepsilon}}', \dot{\boldsymbol{\varepsilon}}^{p'}, \dot{\zeta}'; \zeta) = \frac{2G}{\beta} g(\zeta)\dot{\zeta}' + \mathbb{I}_{\overline{\mathbb{D}}}(\dot{\boldsymbol{\varepsilon}}', \dot{\boldsymbol{\varepsilon}}^{p'}, \dot{\zeta}') \\ &\overline{\mathbb{D}} = \{ (\dot{\boldsymbol{\varepsilon}}', \dot{\boldsymbol{\varepsilon}}^{p'}, \dot{\zeta}') \in \mathbb{V} \quad \text{such that } \mathrm{tr}(\dot{\boldsymbol{\varepsilon}}^{p'}) = 0 \text{ and } \dot{\zeta}' = \|\dot{\boldsymbol{\varepsilon}}^{p'}\| \}\end{aligned} \tag{57}$$

The set $\overline{\mathbb{D}}$ and the function $\tilde{\phi}$ are not convex (see Fig. 6a). However, the Legendre–Fenchel conjugate of $\tilde{\phi}$ is still well-posed (Appendix A, item 5) and can be explicitly derived from the standard procedure:

$$\phi^*(\boldsymbol{\sigma}^{d'}, \boldsymbol{\tau}^{d'}, R^{d'}; \zeta) = \sup_{(\dot{\boldsymbol{\varepsilon}}', \dot{\boldsymbol{\varepsilon}}^{p'}, \dot{\zeta}') \in \overline{\mathbb{D}}} (\boldsymbol{\sigma}^{d'} : \dot{\boldsymbol{\varepsilon}}' + \boldsymbol{\tau}^{d'} : \dot{\boldsymbol{\varepsilon}}^{p'} + R^{d'}\dot{\zeta}' - \tilde{\phi}) = \mathbb{I}_0(\boldsymbol{\sigma}^{d'}) + \mathbb{I}_{\mathbb{E}}(\boldsymbol{\tau}^{d'}, R^{d'}; \zeta) \tag{58}$$

with $\mathbb{E} = \{(\boldsymbol{\tau}^{d'}, R^{d'}) \in \mathbb{S}^2 \times \mathbb{R} \text{ such that } f(\boldsymbol{\tau}^{d'}, R^{d'}; \zeta) \leqslant 0\}$ and

$$f(\boldsymbol{\tau}^{d'}, R^{d'}; \zeta) = \|\mathrm{dev}(\boldsymbol{\tau}^{d'})\| - \frac{2G}{\beta} g(\zeta) + R^{d'} \tag{59}$$

Provided that $\frac{2G}{\beta} = \sqrt{\frac{2}{3}}\sigma_y$, Eqs. (58) and (59) also define the Legendre–Fenchel conjugate of the proper convex lower semi-continuous function (Appendix A, item 5)

$$\begin{aligned}&\phi = \mathrm{cl}(\mathrm{conv}\tilde{\phi}) = \sqrt{\frac{2}{3}}\sigma_y g(\zeta)\dot{\zeta}' + \mathbb{I}_{\overline{\mathbb{D}}}(\dot{\boldsymbol{\varepsilon}}', \dot{\boldsymbol{\varepsilon}}^{p'}, \dot{\zeta}') \\ &\overline{\mathbb{D}} = \left\{ (\dot{\boldsymbol{\varepsilon}}', \dot{\boldsymbol{\varepsilon}}^{p'}, \dot{\zeta}') \in \mathbb{V} \text{ such that } \mathrm{tr}(\dot{\boldsymbol{\varepsilon}}^{p'}) = 0 \text{ and } \dot{\zeta}' \geqslant \|\dot{\boldsymbol{\varepsilon}}^{p'}\| \right\}\end{aligned} \tag{60}$$

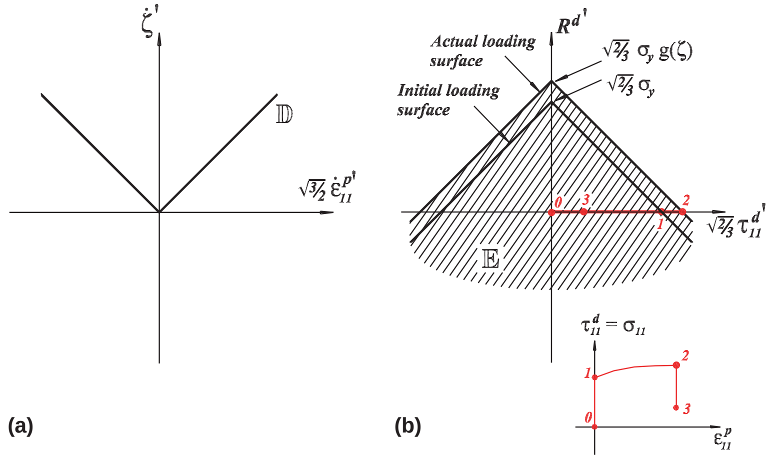

Fig. 6. Endochronic model vs. Prandtl–Reuss model. Tension-compression case. (a) The set $\mathbb{D}$ is the projection of $\overline{\mathbb{D}}$ on the $(\dot{\varepsilon}^{p\prime}, \dot{\zeta}')$-plane, where $\overline{\mathbb{D}}$ is the non-convex effective domain of the pseudo-potential $\tilde{\phi}$ of Eq. (57). It defines an endochronic model where the intrinsic time flow $\dot{\zeta}$ equals the norm of $\dot{\varepsilon}^p$. (b) The convex set $\mathbb{E}$ associated with the indicator function ϕ^* given in Eqs. (58) and (59), which is the Legendre–Fenchel conjugated of $\tilde{\phi}$.

which corresponds to the pseudo-potential of a modified Prandtl–Reuss model, in the case $\xi(\zeta) = \frac{\mathrm{d}\xi(\zeta)}{\mathrm{d}\zeta} = 0$ (see Eqs. (32)).

A similar comparison between the classical Prandtl–Reuss model (Eqs. (23) and (25)) and endochronic models is possible as well, but only when the former is perfectly plastic, i.e. if $\xi(\zeta) = 0$, and conditions $\xi(\zeta) = 0$ and $g = 1$ hold in the latters. Note that these assumptions have been adopted in Fig. 5.

4.3. Multi-layer models of endochronic type

The concept of assembling in parallel several plastic elements can be applied to the case in which each element is of endochronic type. The approach is analogous to the one adopted in Section 3.4. Let $\boldsymbol{\varepsilon}$ and $(\boldsymbol{\varepsilon}^p_i, \zeta_i)$ be the relevant state variables. Then, the Helmholtz energy is defined as the sum of N contributions, of the same kind as in Eq. (50):

$$\Psi = \sum_{i=1}^{N} \Psi_i = \sum_{i=1}^{N} \left[\frac{1}{2} (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p_i) : \mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p_i) \right] \tag{61}$$

where the internal variables $\boldsymbol{\varepsilon}^p_i$ have the meaning of plastic strain of the i-th endochronic element. The thermodynamic forces associated with ζ_i are zero, viz. $R^{nd}_i = 0$. Moreover, N *independent* pseudo-potentials are assumed to be of the type (51):

$$\begin{aligned} \phi_i &= \frac{\|\operatorname{dev}[\mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p_i)]\|^2}{2G g_i(\zeta_i)/\beta_i} \dot{\zeta}'_i + \mathbb{1}_{\overline{\mathbb{D}}_i}(\dot{\boldsymbol{\varepsilon}}', \dot{\boldsymbol{\varepsilon}}^{p\prime}_i, \dot{\zeta}'_i; \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p_i, \zeta_i) \\ \overline{\mathbb{D}}_i &= \left\{ \begin{array}{l} (\dot{\boldsymbol{\varepsilon}}', \dot{\boldsymbol{\varepsilon}}^{p\prime}_i, \dot{\zeta}'_i) \in \mathbb{V} \quad \text{such that} \\ \operatorname{tr}(\dot{\boldsymbol{\varepsilon}}^{p\prime}_i) = 0, \quad \dot{\zeta}'_i \geqslant 0 \text{ and } \dot{\boldsymbol{\varepsilon}}^{p\prime}_i = \dfrac{\operatorname{dev}[\mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p_i)]}{2G g_i(\zeta_i)/\beta_i} \dot{\zeta}'_i \end{array} \right\} \end{aligned} \tag{62}$$

with $\beta_i > 0$, $g_i(\zeta_i) > 0$ and $g_i(0) = 1$. The pseudo-potential of the multi-layer model is $\phi = \sum_{i=1}^{N} \phi_i$ and its dual is $\phi^* = \sum_{i=1}^{N} \phi^*_i$, with

$$\phi_i^* = \sup_{(\dot{\varepsilon}', \dot{\varepsilon}_i^{p'}, \dot{\zeta}_i') \in \overline{\mathbb{D}}_i} (\boldsymbol{\sigma}^{d'} : \dot{\varepsilon}' + \boldsymbol{\tau}_i^{d'} : \dot{\varepsilon}_i^{p'} + R_i^d \dot{\zeta}_i' - \phi_i) = \mathbb{I}_0(\boldsymbol{\sigma}^{d'}) + \mathbb{I}_{\mathbb{E}_i}(\boldsymbol{\tau}_i^{d'}, R_i^{d'}; \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_i^p, \zeta_i) \quad (63)$$

where $\mathbb{E}_i = \{(\boldsymbol{\tau}_i^{d'}, R_i^{d'}) \in \mathbb{S}^2 \times \mathbb{R}$ such that $f_i(\boldsymbol{\tau}_i^{d'}, R_i^{d'}; \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_i^p, \zeta_i) \leqslant 0\}$ and

$$f_i = \frac{\operatorname{dev}(\boldsymbol{\tau}_i^{d'}) : \operatorname{dev}[\mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_i^p)]}{2G g_i(\zeta_i)/\beta_i} - \frac{\|\operatorname{dev}[\mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_i^p)]\|^2}{2G g_i(\zeta_i)/\beta_i} + R_i^{d'} \quad (64)$$

The flow rules then become of the form (55). Moreover, it can be easily proved that at the actual state represented by $(\boldsymbol{\tau}_i^d, R_i^d)$, the identities $f_i = \dot{f}_i = 0$ hold and, for this reason, the fluxes $\dot{\zeta}_i = \dot{\lambda}_i \geqslant 0$ cannot be computed from the consistency conditions and have to be defined using a further assumption.

If the number of elements is $N = 2$, $g_1 = g_2 = 1$ and both fluxes $\dot{\zeta}_1$ and $\dot{\zeta}_2$ are of the form (56), then the model of Casciati (1989) is retrieved. Moreover, the condition $\dot{\zeta}_i' = \|\dot{\boldsymbol{\varepsilon}}_i^{p'}\|$ into (62) leads to a multi-layer model of Prandtl–Reuss type (see Section 4.2).

5. Non-linear kinematic hardening models

The NLK hardening rule was first suggested by Armstrong and Frederick (1966), who introduced a *dynamic recovery* term in the classical Prager's linear kinematic hardening rule. Several modifications of this basic rule have been proposed, in order to improve the description of the cyclic behavior of metals, particularly for the ratchetting phenomenon (see, among others, Chaboche, 1991; Ohno and Wang, 1993).

According to traditional formulation, NLK hardening models *do not* fulfil the assumption of generalized normality (Lemaitre and Chaboche, 1990, pp. 219–221; Chaboche et al., 1995). Following an approach based on the notion of bipotential, De Saxcé (1992) introduced *implicit standard materials* and showed that the plasticity models with NLK hardening rules are of such type.

In this section, another formulation is suggested, which leads to the proof that NLK hardening models belong to the class of generalized standard materials, provided that a suitable, non-conventional, loading function is defined. First, the state variables $\mathbf{v} = (\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \zeta, \boldsymbol{\beta}, \zeta_1)$ have to be introduced. The first three are the same as for Prandtl–Reuss and endochronic models, while $\boldsymbol{\beta}$ and ζ_1 are related to NLK hardening rule. The role of the scalar variable ζ_1 will be discussed later on. The corresponding thermodynamic forces are $\mathbf{q}^{nd} = (\boldsymbol{\sigma}^{nd}, \boldsymbol{\tau}^{nd}, R^{nd}, \mathbf{X}^{nd}, R_1^{nd})$ and $\mathbf{q}^d = (\boldsymbol{\sigma}^d, \boldsymbol{\tau}^d, R^d, \mathbf{X}^d, R_1^d) \in \mathbb{V}^*$. The Helmholtz energy density is chosen as follows:

$$\Psi = \frac{1}{2}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) : \mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) + \frac{1}{2}(\boldsymbol{\varepsilon}^p - \boldsymbol{\beta}) : \mathbf{D} : (\boldsymbol{\varepsilon}^p - \boldsymbol{\beta}) \quad (65)$$

The quantity $\boldsymbol{\alpha} = \boldsymbol{\varepsilon}^p - \boldsymbol{\beta}$ is usually adopted as the internal variable associated with the kinematic hardening. However, the choice of $\boldsymbol{\beta}$ as a representative internal variable appears more suited, because it highlights the formal analogy between the first quadratic term in Eq. (65), typical of plasticity models, and the second one, associated with the kinematic hardening. The isotropy assumption leads to the usual expression for $\mathbf{C}$ and entails that $\mathbf{D} = D_1 \mathbf{1} \otimes \mathbf{1} + D_2 \mathbf{I}$. The non-dissipative forces can then be readily evaluated:

$$\begin{aligned}
\boldsymbol{\sigma}^{nd} &= \mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) \\
\boldsymbol{\tau}^{nd} &= -\mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) + \mathbf{D} : (\boldsymbol{\varepsilon}^p - \boldsymbol{\beta}), \quad R^{nd} = 0 \\
\mathbf{X}^{nd} &= -\mathbf{D} : (\boldsymbol{\varepsilon}^p - \boldsymbol{\beta}), \quad R_1^{nd} = 0
\end{aligned} \quad (66)$$

The three tensorial non-dissipative forces are related by the identity $\boldsymbol{\tau}^{nd} = -\boldsymbol{\sigma}^{nd} - \mathbf{X}^{nd}$. Moreover, let the pseudo-potential be equal to

$$\phi = \sqrt{\frac{2}{3}}\sigma_y g(\zeta)\dot{\zeta}' + \frac{\|\mathbf{D} : (\boldsymbol{\varepsilon}^p - \boldsymbol{\beta})\|^2}{\frac{D_2}{\delta} g_1(\zeta_1)}\dot{\zeta}_1' + \mathbb{I}_{\overline{\mathbb{D}}}(\dot{\boldsymbol{\varepsilon}}', \dot{\boldsymbol{\varepsilon}}^{p'}, \dot{\zeta}', \dot{\boldsymbol{\beta}}', \dot{\zeta}_1'; \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \zeta, \boldsymbol{\beta}, \zeta_1)$$

$$\overline{\mathbb{D}} = \left\{ \begin{array}{l} (\dot{\boldsymbol{\varepsilon}}', \dot{\boldsymbol{\varepsilon}}^{p'}, \dot{\zeta}', \dot{\boldsymbol{\beta}}', \dot{\zeta}_1') \in \mathbb{V} \quad \text{such that} \\ \mathrm{tr}\,(\dot{\boldsymbol{\varepsilon}}^{p'}) = 0, \quad \dot{\zeta}' \geqslant \|\dot{\boldsymbol{\varepsilon}}^{p'}\|, \\ \mathrm{tr}\,(\dot{\boldsymbol{\beta}}') = 0, \quad \dot{\boldsymbol{\beta}}' = \dfrac{\mathbf{D} : (\boldsymbol{\varepsilon}^p - \boldsymbol{\beta})}{\frac{D_2}{\delta} g_1(\zeta_1)}\ \dot{\zeta}_1', \\ \dot{\zeta}_1' = h(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \zeta, \boldsymbol{\beta}, \zeta_1)\dot{\zeta}' \geqslant 0 \end{array} \right\} \tag{67}$$

with δ, $g(\zeta)$, $g_1(\zeta_1) > 0$ and $g(0) = g_1(0) = 1$. Fig. 7 shows two projections of the effective domain $\overline{\mathbb{D}}$ for the tension-compression case. The first term in the definition of ϕ is identical to that of Eq. (32) for a modified Prandtl–Reuss model with $\xi(\zeta) = 0$. The second term is related to the NLK hardening and *it is formally identical to the one used in the definition of endochronic models* (see Eq. (51)), with the substitutions dev$(\boldsymbol{\varepsilon}) \to \boldsymbol{\varepsilon}^p$, $\boldsymbol{\varepsilon}^p \to \boldsymbol{\beta}$ and $\zeta \to \zeta_1$. The same analogy applies to the conditions defining the set $\overline{\mathbb{D}}$.

The dual pseudo-potential then becomes

$$\begin{aligned} \phi^* &= \sup_{(\dot{\boldsymbol{\varepsilon}}', \dot{\boldsymbol{\varepsilon}}^{p'}, \dot{\zeta}', \dot{\boldsymbol{\beta}}', \dot{\zeta}_1') \in \overline{\mathbb{D}}} \left(\boldsymbol{\sigma}^{d'} : \dot{\boldsymbol{\varepsilon}}' + \boldsymbol{\tau}^{d'} : \dot{\boldsymbol{\varepsilon}}^{p'} + R^{d'}\dot{\zeta}' + \mathbf{X}^{d'} : \dot{\boldsymbol{\beta}}' + R_1^{d'}\dot{\zeta}_1' - \phi \right) \\ &= \mathbb{I}_0(\boldsymbol{\sigma}^{d'}) + \mathbb{I}_{\mathbb{E}}(\boldsymbol{\tau}^{d'}, R^{d'}, \mathbf{X}^{d'}, R_1^{d'}; \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \zeta, \boldsymbol{\beta}, \zeta_1) \end{aligned} \tag{68}$$

where $\mathbb{E} = \{(\boldsymbol{\tau}^{d'}, R^{d'}, \mathbf{X}^{d'}, R_1^{d'}) \in \mathbb{S}^2 \times \mathbb{R} \times \mathbb{S}^2 \times \mathbb{R}$ such that $f \leqslant 0\}$ and

$$f = \|\mathrm{dev}\,(\boldsymbol{\tau}^{d'})\| - \sqrt{\frac{2}{3}}\sigma_y g(\zeta) + R^{d'} + \left(\frac{\mathbf{X}^{d'} : [\mathbf{D} : (\boldsymbol{\varepsilon}^p - \boldsymbol{\beta})]}{D_2 g_1(\zeta_1)/\delta} - \frac{\|\mathbf{D} : (\boldsymbol{\varepsilon}^p - \boldsymbol{\beta})\|^2}{D_2 g_1(\zeta_1)/\delta} + R_1^{d'} \right) h(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \zeta, \boldsymbol{\beta}, \zeta_1) \tag{69}$$

Eq. (69) defines the loading function of a model with NLK hardening and the associated set $\mathbb{E}$ is depicted in Fig. 8 for the tension-compression case when $g(\zeta) = 1$. The normality condition associated with ϕ^* leads to the following flow rules:

$$\begin{aligned} &\dot{\boldsymbol{\varepsilon}}^p = \frac{\mathrm{dev}\,(\boldsymbol{\tau}^d)}{\|\mathrm{dev}\,(\boldsymbol{\tau}^d)\|}\dot{\lambda} = \mathbf{n}\dot{\lambda}, \quad \dot{\zeta} = \dot{\lambda} \\ &\dot{\boldsymbol{\beta}} = \frac{\mathbf{D} : (\boldsymbol{\varepsilon}^p - \boldsymbol{\beta})}{D_2 g_1(\zeta_1)/\delta} h(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \zeta, \boldsymbol{\beta}, \zeta_1)\dot{\lambda}, \quad \dot{\zeta}_1 = h(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \zeta, \boldsymbol{\beta}, \zeta_1)\dot{\lambda} \\ &\quad \text{with } \dot{\lambda} \geqslant 0,\ f \leqslant 0,\ \dot{\lambda} f = 0 \end{aligned} \tag{70}$$

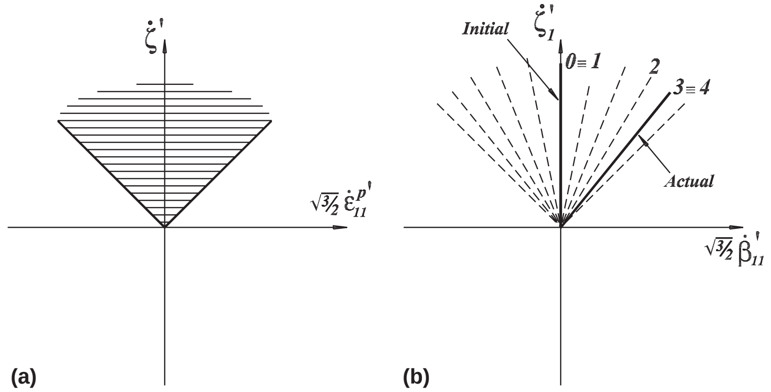

Fig. 7. NLK hardening model. Tension-compression case. (a) Projection of the effective domain $\overline{\mathbb{D}}$ on the $(\dot{\boldsymbol{\varepsilon}}^{p'}, \dot{\zeta}')$-plane. (b) Projection of $\overline{\mathbb{D}}$ on the $(\dot{\boldsymbol{\beta}}', \dot{\zeta}_1')$-plane.

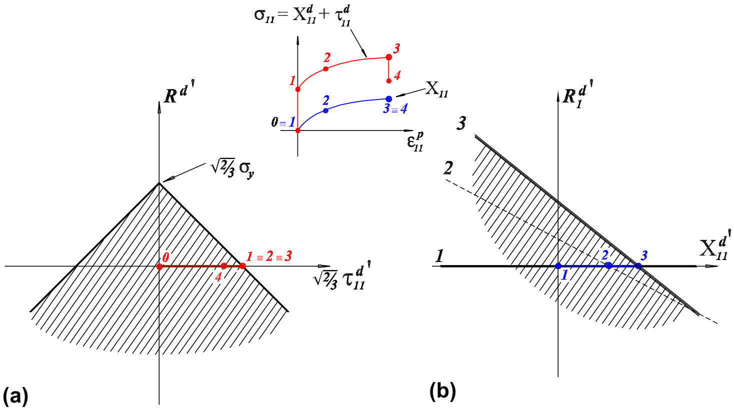

Fig. 8. NLK hardening model. Tension-compression case with $g(\zeta)=1$. (a) Condition $f\leqslant 0$ when $R_1^{d'}=R_1^d=0$ and $\mathbf{X}^{d'}=\mathbf{X}^d=\mathbf{D}:(\boldsymbol{\varepsilon}^p-\boldsymbol{\beta})$. (b) Condition $f\leqslant 0$ when $R^{d'}=R^d=0$ and $\|\mathrm{dev}(\boldsymbol{\tau}^{d'})\|=\|\mathrm{dev}(\boldsymbol{\tau}^d)\|=\sqrt{\frac{2}{3}}\sigma_y$.

The thermodynamic force $\mathbf{X}^d=-\mathbf{X}^{nd}=\mathbf{D}{:}(\boldsymbol{\varepsilon}^p-\boldsymbol{\beta})$ is traceless, due to the assumptions adopted for the traces of $\dot{\boldsymbol{\varepsilon}}^p$ and $\dot{\boldsymbol{\beta}}$. Special attention must be paid to the relationship between the fluxes $\dot{\zeta}_1$ and $\dot{\zeta}$. The time derivative of ζ_1 is defined as the product between $\dot{\zeta}$ and the function h, which depends on the state variables and must be non-negative and finite, but is otherwise free. The variable ζ_1 can be interpreted as an *intrinsic time scale for the NLK hardening flow rule*.

Accounting for the identities $(\boldsymbol{\tau}^{nd},R^{nd},\mathbf{X}^{nd},R_1^{nd})=-(\boldsymbol{\tau}^d,R^d,\mathbf{X}^d,R_1^d)$ and Eqs. (66), one can prove that $R^d=0$ and that the term proportional to h in Eq. (69) is always zero at the actual state. Hence, only the first two terms in the expression of f affect the consistency condition $\dot{f}=0$, which leads to the plastic multiplier

$$\dot{\lambda}=H(f)\frac{\langle\mathbf{n}:\dot{\boldsymbol{\varepsilon}}\rangle}{1+\frac{D_2}{2G}-\frac{1}{2G}\frac{\mathbf{n}:\mathbf{X}^d}{g_1(\zeta_1)/\delta}h(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^p,\zeta,\boldsymbol{\beta},\zeta_1)+\sqrt{\frac{2}{3}}\frac{\sigma_y}{2G}\frac{\mathrm{d}g(\zeta)}{\mathrm{d}\zeta}} \tag{71}$$

The positive functions g, g_1 and h determine the actual model.

The choice $g=g_1=h=1$ corresponds to the basic NLK hardening model of Armstrong and Frederick (1966). Another interesting case is given by $g=g_1=1$ and

$$\begin{cases} h=\left(\dfrac{\|\mathbf{D}:(\boldsymbol{\varepsilon}^p-\boldsymbol{\beta})\|}{D_2/\delta}\right)^{m_1}\langle\mathbf{k}_1:\mathbf{n}\rangle & \text{if } \mathbf{D}:(\boldsymbol{\varepsilon}^p-\boldsymbol{\beta})\neq\mathbf{0} \\ h=0 & \text{if } \mathbf{D}:(\boldsymbol{\varepsilon}^p-\boldsymbol{\beta})=\mathbf{0} \end{cases} \tag{72}$$

where $m_1>0$ and $\mathbf{k}_1=\frac{\mathbf{D}:(\boldsymbol{\varepsilon}^p-\boldsymbol{\beta})}{\|\mathbf{D}:(\boldsymbol{\varepsilon}^p-\boldsymbol{\beta})\|}$ is the unit vector having the same direction as $\mathbf{X}^d=\mathbf{D}:(\boldsymbol{\varepsilon}^p-\boldsymbol{\beta})$. These conditions lead to

$$\dot{\boldsymbol{\beta}}=\frac{\mathbf{X}^d}{D_2/\delta}\dot{\zeta}_1=\frac{\mathbf{X}^d}{D_2/\delta}\left(\frac{\|\mathbf{X}^d\|}{D_2/\delta}\right)^{m_1}\langle\mathbf{k}_1:\dot{\boldsymbol{\varepsilon}}^p\rangle=\dot{\boldsymbol{\varepsilon}}^p-\frac{\dot{\mathbf{X}}^d}{D_2} \tag{73}$$

which is the NLK hardening rule proposed by (Ohno and Wang, 1993) for modelling the ratchetting phenomenon in metal plasticity. It is interesting to compare the quantity

$$\dot{\zeta}_1=h\dot{\zeta}=\left(\frac{\|\mathbf{X}^d\|}{D_2/\delta}\right)^{m_1}\langle\mathbf{k}_1:\dot{\boldsymbol{\varepsilon}}^p\rangle \tag{74}$$

and the intrinsic time flow $\dot{\vartheta}$, defined in Eq. (56) for endochronic models of the Bouc-Wen type. Two significant differences can be observed: (i) the governing flow variable is the plastic strain for NLK hardening rule and the total strain for the flow rule of the endochronic model; (ii) due to presence of the *absolute value* instead of the McCauley brackets, the endochronic model of Bouc-Wen type introduces non-zero flows during unloading phases when $\gamma \neq \beta$.

5.1. From an endochronic model to a NLK hardening model

Valanis (1980) and Watanabe and Atluri (1986) proved that a NLK hardening model can be derived from the endochronic theory by adopting a special intrinsic-time definition, namely when the intrinsic time scale flow $\dot{\zeta}$ is forced to be equal to the norm of the plastic strain flow. The approach suggested in this paper not only confirms this result, but allows for a generalization, due to the presence of a second intrinsic time scale ζ_1, in general distinct from ζ. Consider the differential equations defining an endochronic model with a *kinematic hardening* variable $\mathbf{X}^d$:

$$\begin{cases} \mathrm{tr}\,(\dot{\boldsymbol{\sigma}}) = 3K\mathrm{tr}\,(\dot{\boldsymbol{\varepsilon}}) \\ \mathrm{dev}\,(\dot{\boldsymbol{\sigma}}) = 2G\,\mathrm{dev}\,(\dot{\boldsymbol{\varepsilon}}) - \beta\,\mathrm{dev}\,(\boldsymbol{\sigma} - \mathbf{X}^d)\frac{\dot{\zeta}}{g(\zeta)} \\ \mathrm{tr}\,(\dot{\mathbf{X}}^d) = 0 \\ \dot{\mathbf{X}}^d = D_2\dot{\boldsymbol{\varepsilon}}^p - \delta\mathbf{X}^d\frac{\dot{\zeta}_1}{g_1(\zeta_1)} \\ \dot{\zeta}_1 = h(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \zeta, \boldsymbol{\beta}, \zeta_1)\dot{\zeta} \end{cases} \tag{75}$$

The idea of a kinematic hardening variable in an endochronic model was first suggested by Bažant (1978), who however considered a *linear* evolution of X^d as function of the plastic strain. An alternative way to describe the model defined by (75) is

$$\begin{cases} \boldsymbol{\sigma} = \mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) & \mathbf{X}^d = \mathbf{D} : (\boldsymbol{\varepsilon}^p - \boldsymbol{\beta}) \\ \mathbf{C} = \left(K - \frac{2}{3}G\right)\mathbf{1}\otimes\mathbf{1} + 2G\mathbf{I} & \mathbf{D} = D_1\mathbf{1}\otimes\mathbf{1} + D_2\mathbf{I} \\ \mathrm{tr}\,(\dot{\boldsymbol{\varepsilon}}^p) = 0,\ \ \dot{\boldsymbol{\varepsilon}}^p = \dfrac{\mathrm{dev}\,(\boldsymbol{\sigma} - \mathbf{X}^d)}{\frac{2G}{\beta}g(\zeta)}\dot{\zeta} & \mathrm{tr}\,(\dot{\boldsymbol{\beta}}) = 0,\ \ \dot{\boldsymbol{\beta}} = \dfrac{\mathbf{X}^d}{\frac{D_2}{\delta}g_1(\zeta_1)}\dot{\zeta}_1 \\ & \dot{\zeta}_1 = h(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \zeta, \boldsymbol{\beta}, \zeta_1)\dot{\zeta} \end{cases} \tag{76}$$

Moreover, both Eqs. (75) and (76) can be derived from (65) and the following pseudo-potential:

$$\phi = \frac{\|\mathrm{dev}\,[\mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) - \mathbf{D} : (\boldsymbol{\varepsilon}^p - \boldsymbol{\beta})]\|^2}{\frac{2G}{\beta}g(\zeta)}\dot{\zeta}' + \frac{\|\mathbf{D} : (\boldsymbol{\varepsilon}^p - \boldsymbol{\beta})\|^2}{\frac{D_2}{\delta}g_1(\zeta_1)}\dot{\zeta}_1' + \mathbb{I}_{\overline{\mathbb{D}}}(\dot{\boldsymbol{\varepsilon}}', \dot{\boldsymbol{\varepsilon}}^{p'}, \dot{\zeta}', \dot{\boldsymbol{\beta}}', \dot{\zeta}_1'; \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \zeta, \boldsymbol{\beta}, \zeta_1)$$

$$\overline{\mathbb{D}} = \left\{ \begin{array}{l} (\dot{\boldsymbol{\varepsilon}}', \dot{\boldsymbol{\varepsilon}}^{p'}, \dot{\zeta}', \dot{\boldsymbol{\beta}}', \dot{\zeta}_1') \in \mathbb{V} \quad \text{such that } \mathrm{tr}\,(\dot{\boldsymbol{\varepsilon}}^{p'}) = 0, \quad \dot{\boldsymbol{\varepsilon}}^{p'} = \dfrac{\mathrm{dev}\,[\mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) - \mathbf{D} : (\boldsymbol{\varepsilon}^p - \boldsymbol{\beta})]}{\frac{2G}{\beta}g(\zeta)}\dot{\zeta}', \quad \dot{\zeta}' \geqslant 0, \\ \mathrm{tr}\,(\dot{\boldsymbol{\beta}}') = 0, \quad \dot{\boldsymbol{\beta}}' = \dfrac{\mathbf{D} : (\boldsymbol{\varepsilon}^p - \boldsymbol{\beta})}{\frac{D_2}{\delta}g_1(\zeta_1)}\dot{\zeta}_1', \quad \dot{\zeta}_1' = h(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \zeta, \boldsymbol{\beta}, \zeta_1)\dot{\zeta}' \geqslant 0 \end{array} \right\} \tag{77}$$

Let $\dot{\zeta}' = \|\dot{\boldsymbol{\varepsilon}}^{p'}\|$ be the chosen intrinsic time definition and assume $\frac{2G}{\beta} = \sqrt{\frac{2}{3}}\sigma_y$. Then, introducing these conditions in (77), one obtains a pseudo-potential $\tilde{\phi}$ which differs from the one of Eq. (67) only in the inequality $\dot{\zeta}' \geqslant \|\dot{\boldsymbol{\varepsilon}}^{p'}\|$, which is an equality in $\tilde{\phi}$. This difference affects neither the expression of the dual pseudo-potential $\tilde{\phi}^* = \phi^*$ (Appendix A, item 6) nor the flow rules, which are in both cases equal to Eqs. (68)–(69) and Eq. (70), respectively. Moreover, in the particular case $h = 1$ and $g(\zeta) = g_1(\zeta)$, the results discussed by Valanis (1980) and Watanabe and Atluri (1986) are retrieved.

6. Generalized plasticity models

Generalized plasticity models (Lubliner et al., 1993) are considered an effective alternative to NLK hardening models, since they behave similarly and are computationally less expensive (Auricchio and Taylor, 1995). A new description of these models is suggested here, supported by a suitable pseudo-potential and the generalized normality assumption. In order to expose the basic principles of this new approach, only the simple generalized plasticity model presented by Auricchio and Taylor (1995) is considered. The extension to more complex cases is straightforward.

First, the state variables $\mathbf{v} = (\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \zeta)$ have to be introduced. The corresponding thermodynamic forces are $\mathbf{q}^{nd} = (\boldsymbol{\sigma}^{nd}, \boldsymbol{\tau}^{nd}, R^{nd})$ and $\mathbf{q}^d = (\boldsymbol{\sigma}^d, \boldsymbol{\tau}^d, R^d)$. The Helmholtz energy density is chosen as follows:

$$\Psi = \frac{1}{2}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) : \mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) + \frac{1}{2}\boldsymbol{\varepsilon}^p : \mathbf{D} : \boldsymbol{\varepsilon}^p \tag{78}$$

The expression for $\mathbf{C}$ and $\mathbf{D}$ are the same as in NLK hardening models. The non-dissipative forces can be readily evaluated:

$$\boldsymbol{\sigma}^{nd} = \mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p), \quad \boldsymbol{\tau}^{nd} = -\mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) + \mathbf{D} : \boldsymbol{\varepsilon}^p, \quad R^{nd} = 0 \tag{79}$$

Note that $\boldsymbol{\sigma}^{nd}$ and $\boldsymbol{\tau}^{nd}$ are related by the identity $\boldsymbol{\tau}^{nd} = -(\boldsymbol{\sigma}^{nd} - \mathbf{D}:\boldsymbol{\varepsilon}^p)$, where the *backstress* $\mathbf{D}:\boldsymbol{\varepsilon}^p$ introduces a linear kinematic hardening effect. Moreover, let the pseudo-potential be equal to

$$\begin{aligned} &\phi(\dot{\boldsymbol{\varepsilon}}', \dot{\boldsymbol{\varepsilon}}^{p\prime}, \dot{\zeta}'; \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \zeta) = \bar{g}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \zeta)\dot{\zeta}' + \mathbb{I}_{\overline{\mathbb{D}}}(\dot{\boldsymbol{\varepsilon}}', \dot{\boldsymbol{\varepsilon}}^{p\prime}, \dot{\zeta}') \\ &\overline{\mathbb{D}} = \{ (\dot{\boldsymbol{\varepsilon}}', \dot{\boldsymbol{\varepsilon}}^{p\prime}, \dot{\zeta}') \in \mathbb{V} \text{ such that } \operatorname{tr}(\dot{\boldsymbol{\varepsilon}}^{p\prime}) = 0 \text{ and } \dot{\zeta}' \geqslant \|\dot{\boldsymbol{\varepsilon}}^{p\prime}\| \} \end{aligned} \tag{80}$$

where

$$\begin{aligned} &\bar{g}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \zeta) = \begin{cases} \sqrt{\frac{2}{3}}\sigma_y + H_{\text{iso}}\zeta & \text{if } \bar{f} < 0 \\ \|\operatorname{dev}[\mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) - \mathbf{D} : \boldsymbol{\varepsilon}^p]\| & \text{if } \bar{f} \geqslant 0 \end{cases} \\ &\bar{f}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \zeta) := \|\operatorname{dev}[\mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) - \mathbf{D} : \boldsymbol{\varepsilon}^p]\| - \left(\sqrt{\frac{2}{3}}\sigma_y + H_{\text{iso}}\zeta\right) \end{aligned} \tag{81}$$

with $H_{\text{iso}} \geqslant 0$. The main characteristic of this pseudo-potential function is given by the *piecewise* expression introduced to define the positive function $\bar{g}$. It is assumed that $\bar{g}$ depends on the sign of the function $\bar{f}$, which in turn is related to the state variables. The conjugated pseudo-potential ϕ^* reads

$$\phi^*(\boldsymbol{\sigma}^{d\prime}, \boldsymbol{\tau}^{d\prime}, R^{d\prime}; \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \zeta) = \sup_{(\dot{\boldsymbol{\varepsilon}}', \dot{\boldsymbol{\varepsilon}}^{p\prime}, \dot{\zeta}') \in \overline{\mathbb{D}}} (\boldsymbol{\sigma}^{d\prime} : \dot{\boldsymbol{\varepsilon}}' + \boldsymbol{\tau}^{d\prime} : \dot{\boldsymbol{\varepsilon}}^{p\prime} + R^{d\prime}\dot{\zeta}' - \phi) = \mathbb{I}_0(\boldsymbol{\sigma}^{d\prime}) + \mathbb{I}_{\mathbb{E}}(\boldsymbol{\tau}^{d\prime}, R^{d\prime}; \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \zeta) \tag{82}$$

where $\mathbb{E} = \{(\boldsymbol{\tau}^{d\prime}, R^{d\prime}) \in \mathbb{S}^2 \times \mathbb{R} \text{ such that } f(\boldsymbol{\tau}^{d\prime}, R^{d\prime}; \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \zeta) \leqslant 0\}$ and

$$f(\boldsymbol{\tau}^{d\prime}, R^{d\prime}; \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \zeta) = \begin{cases} \|\operatorname{dev}(\boldsymbol{\tau}^{d\prime})\| - \left(\sqrt{\frac{2}{3}}\sigma_y + H_{\text{iso}}\zeta\right) + R^{d\prime} & \text{if } \bar{f} < 0 \\ \|\operatorname{dev}(\boldsymbol{\tau}^{d\prime})\| - \|\operatorname{dev}[\mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) - \mathbf{D} : \boldsymbol{\varepsilon}^p]\| + R^{d\prime} & \text{if } \bar{f} \geqslant 0 \end{cases} \tag{83}$$

The loading function f also has a twofold definition: recalling that the actual thermodynamic force $\boldsymbol{\tau}^d$ fulfils the following identities

$$\boldsymbol{\tau}^d = \mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) - \mathbf{D} : \boldsymbol{\varepsilon}^p = \boldsymbol{\sigma} - \mathbf{D} : \boldsymbol{\varepsilon}^p \tag{84}$$

and $R^d = -R^{nd} = 0$, one can prove that if $\bar{f}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \zeta) < 0$ then $f(\boldsymbol{\tau}^d, R^d; \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \zeta) = \bar{f}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \zeta)$; moreover, if $\bar{f}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \zeta) \geqslant 0$, then $f(\boldsymbol{\tau}^d, R^d; \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \zeta)$ is always zero, viz. the actual state represented by $(\boldsymbol{\tau}^d, R^d)$ remains in contact with the loading surface $\partial\mathbb{E}$. In Fig. 9, this situation is depicted for the tension-compression case.

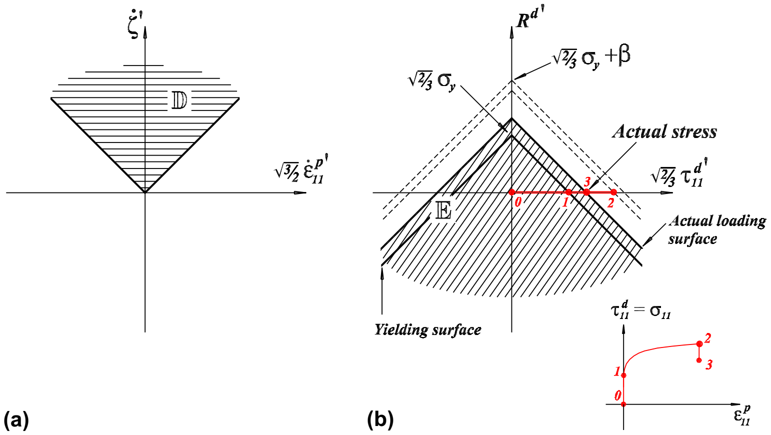

Fig. 9. Generalized plasticity. Tension-compression case. (a) Projection of the pseudo-potential effective domain $\overline{\mathbb{D}}$ on the $(\dot{\boldsymbol{\varepsilon}}^{p'},\dot{\zeta}')$-plane. This set is indicated by $\mathbb{D}$. (b) Several configurations of the domain $\mathbb{E}$. When $\bar{f}\geqslant 0$, $\mathbb{E}$ translates upward during loading phases and downward during unloading phases. The point $(\boldsymbol{\tau}^d, R^d)$, representing the actual state, always lies on the axis $R^{d'}=0$.

The normality conditions associated with the loading function f read:

$$\dot{\boldsymbol{\varepsilon}}^p=\frac{\operatorname{dev}(\boldsymbol{\tau}^d)}{\|\operatorname{dev}(\boldsymbol{\tau}^d)\|}\dot{\lambda}=\mathbf{n}\dot{\lambda},\quad \dot{\zeta}=\dot{\lambda}\quad \text{with } \dot{\lambda}f=0,\ f\leqslant 0,\ \dot{\lambda}\geqslant 0 \tag{85}$$

These flow rules are identical to those of a Prandtl–Reuss model (see Eqs. (35)). However, they derive from a different loading function and for this reason the computation of the plastic multiplier $\dot{\lambda}$ is not the same. When $f(\boldsymbol{\tau}^d,R^d;\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^p,\zeta)=\bar{f}(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^p,\zeta)<0$, the loading–unloading conditions reduce to $\dot{\lambda}=0$, leading to an elastic behavior. As a result, the function $\bar{f}$ is also called *yielding function*, while the surface defined by the condition $\bar{f}=0$ is called *yielding surface*. Conversely, when $\bar{f}\geqslant 0$ the set $\mathbb{E}$ evolves by virtue of the dependence of f on the state variables $\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon}^p$ and ζ. During this evolution, the actual thermodynamic forces $(\boldsymbol{\tau}^d,R^d)$ always satisfy the condition $f=0$. Moreover, the consistency condition

$$\dot{f}=\left[\frac{\partial f}{\partial\boldsymbol{\tau}^{d'}}:\dot{\boldsymbol{\tau}}^{d'}+\frac{\partial f}{\partial R^{d'}}\dot{R}^{d'}+\frac{\partial f}{\partial\boldsymbol{\varepsilon}}:\dot{\boldsymbol{\varepsilon}}+\frac{\partial f}{\partial\boldsymbol{\varepsilon}^p}:\dot{\boldsymbol{\varepsilon}}^p+\frac{\partial f}{\partial\zeta}\dot{\zeta}\right]_{(\boldsymbol{\tau}^{d'}=\boldsymbol{\tau}^d,R^{d'}=R^d)}=0 \tag{86}$$

is also identically fulfilled and, like for the endochronic theory, it does not permit to compute $\dot{\lambda}\geqslant 0$. Hence, the condition that the so-called *limit function* is equal to zero has to be invoked and this leads to (Auricchio and Taylor, 1995):

$$\dot{\lambda}=\dot{\zeta}=\begin{cases}0 & \text{if } \bar{f}<0\\ \dfrac{\langle\mathbf{n}:\dot{\boldsymbol{\varepsilon}}\rangle}{1+\dfrac{\overline{N}(\overline{M}-\bar{f})+(D_2+H_{\text{iso}})\overline{M}}{2G\bar{f}}} & \text{if } 0\leqslant\bar{f}\leqslant\overline{M}\end{cases} \tag{87}$$

where $\overline{M},\overline{N}>0$. It can be proved that when $\bar{f}$ tends to $\overline{M}$, the expression of the plastic multiplier of a classical plasticity model with linear kinematic and isotropic hardening is retrieved. Moreover, if $H_{\text{iso}}=0$ an asymptotic value of $\|\boldsymbol{\tau}^d\|$ exists, and is equal to $\sqrt{\frac{2}{3}}\sigma_y+\overline{M}$.

7. Conclusions

A common theoretical framework between Prandtl–Reuss models and endochronic theory as well as NLK hardening and generalized plasticity models was constructed. All models were defined assuming generalized normality. It was therefore proved that a unique mathematical structure, based on the notions of pseudo-potential and generalized normality, was able to contain plasticity models traditionally formulated by other approaches. In particular, no extension of the generalized standard class of materials had to be introduced to describe NLK hardening and generalized plasticity models. This approach allowed several comparisons, that have clarified the relationships and analogies between these, a priori different, plasticity theories.

Appendix A

The vector spaces considered in this paper are: (i) the space of second order tensors; (ii) the space of *symmetric* second order tensors $\mathbb{S}^2$; (iii) the set of real scalars $\mathbb{R} = (-\infty, +\infty)$; (iv) the cartesian product of a finite number of such spaces. They are all equipped with an Euclidian product, so they are always isomorph to the Euclidian vector space $\mathbb{X} = \mathbb{R}^n$.

(1) A subset $\mathbb{C}$ of $\mathbb{X}$ is said to be:
  (a) a *convex* set if $(1-\lambda)\mathbf{x} + \lambda\mathbf{y} \in \mathbb{C}$ whenever $\mathbf{x}, \mathbf{y} \in \mathbb{C}$ and $0 < \lambda < 1$.
  (b) a *cone* if $\lambda\mathbf{y} \in \mathbb{C}$ when $\mathbf{y} \in \mathbb{C}$ and $\lambda > 0$.
(2) Let $\phi : \mathbb{X} \to (-\infty, \infty]$ be an extended-real-valued function defined on the vector space $\mathbb{X}$. Then,
  (a) the *epigraph* of ϕ is the set

$$epi\phi = \{(\mathbf{y}, \mu) \text{ such that } \mathbf{y} \in \mathbb{X}, \mu \in \mathbb{R}, \mu \geqslant \phi(\mathbf{y})\} \tag{A.1}$$

  (b) ϕ is said to be *convex on* $\mathbb{X}$ if *epi* ϕ is convex as a subset of $\mathbb{X} \times \mathbb{R}$.
  (c) a convex function ϕ is said to be *proper* if and only if the set

$$\overline{\mathbb{D}} = \{\mathbf{y} \in \mathbb{X} : \phi(\mathbf{y}) < +\infty\} \tag{A.2}$$

  is not empty. The set $\overline{\mathbb{D}}$ is called *effective domain of* ϕ, it is convex since ϕ is convex and is the set where ϕ is finite.
  (d) ϕ is said to be *continuous relative to a set* $\overline{\mathbb{D}}$ if the restriction of ϕ to $\overline{\mathbb{D}}$ is a continuous function.
  (e) ϕ is *lower semicontinuous* at $\mathbf{x} \in \mathbb{X}$ if

$$\phi(\mathbf{x}) = \lim_{\mathbf{y}\to\mathbf{x}} \inf \phi(\mathbf{y}) \tag{A.3}$$

  It can be proved that the condition of lower semi-continuity of ϕ is equivalent to have that the *level set* $\{\mathbf{y}:\phi(\mathbf{y}) \leqslant \alpha\}$ is closed in $\mathbb{X}$ for every $\alpha \in \mathbb{R}$ (Rockafellar, 1969, p. 51). As a result, when ϕ is a proper convex function with a (convex) effective domain $\overline{\mathbb{D}}$ *closed* in $\mathbb{X}$ and ϕ is continuous relative to $\overline{\mathbb{D}}$, then ϕ is lower-semicontinuous (Rockafellar, 1969, p. 52).
(3) Let $\mathbb{X}^*$ be the dual of $\mathbb{X}$. Since $\mathbb{X} = \mathbb{R}^n$, then $\mathbb{X}^{**} = \mathbb{X}$ and the duality product between $\mathbf{x}$ and $\mathbf{x}^*$, elements of the dual vector spaces $\mathbb{X}$ and $\mathbb{X}^*$, can be written as $\mathbf{x}^* \cdot \mathbf{x}$.
  Let $\phi : \mathbb{X} \to (-\infty, \infty]$ be an extended-real-valued *convex* function. Then, the *subgradients* of ϕ at $\mathbf{x} \in \mathbb{X}$ are elements $\mathbf{x}^* \in \mathbb{X}^*$ such that

$$\forall \mathbf{y} \in \mathbb{X}, \quad \phi(\mathbf{y}) - \phi(\mathbf{x}) \geqslant \mathbf{x}^* \cdot (\mathbf{y} - \mathbf{x}) \tag{A.4}$$

  The *subdifferential set* $\partial\phi(\mathbf{x})$ is the set of all subgradients $\mathbf{x}^*$ at $\mathbf{x}$:

$$\partial\phi(\mathbf{x}) = \{\mathbf{x}^* \in \mathbb{X}^* \text{ such that the condition (A.4) holds}\} \tag{A.5}$$

  The function ϕ is said to be *subdifferentiable* at $\mathbf{x}$ when $\partial\phi(\mathbf{x})$ is non-empty.

(4) If a function $\phi : \mathbb{X} \to (-\infty, \infty]$ is *convex, proper, non-negative* and such that $\phi(\mathbf{0}) = 0$, then the *normality* condition

$$\mathbf{x}^* \in \partial\phi(\mathbf{x}) \tag{A.6}$$

viz. $\mathbf{x}^*$ belongs to the subdifferential set of ϕ at $\mathbf{x}$, entails that $\mathbf{x}^* \cdot \mathbf{x} \geqslant 0$.

Proof. Setting $\mathbf{y} = \mathbf{0}$ in the inequality (A.4) entails that, for any $\mathbf{x}$ in the effective domain of ϕ, $-\phi(\mathbf{x}) \geqslant \mathbf{x}^* \cdot (\mathbf{0} - \mathbf{x})$. Hence, by virtue of the non-negativity of ϕ, $\mathbf{x}^* \cdot \mathbf{x} \geqslant 0$.

(5) When a function $\phi : \mathbb{X} \to (-\infty, \infty]$ is proper, convex and lower semi-continuous, the dual function $\phi^* : \mathbb{X}^* \to (-\infty, \infty]$, defined by the *Legendre–Fenchel transform*

$$\forall \mathbf{y}^* \in \mathbb{X}^* \qquad \phi^*(\mathbf{y}^*) = \sup_{\mathbf{y}\in\mathbb{X}} (\mathbf{y}^* \cdot \mathbf{y} - \phi(\mathbf{y})) \tag{A.7}$$

is related to ϕ by a *one-to-one correspondence*, in the sense that for such a kind of functions, the conjugate ϕ^* is in turn proper, convex and lower semi-continuous and $\phi^{**} = \phi$ (Rockafellar, 1969, p. 104). Under these assumptions, it also holds:

$$\forall \mathbf{y}^* \in \mathbb{X}^* \qquad \phi^*(\mathbf{y}^*) = \sup_{\mathbf{y}\in\overline{\mathbb{D}}} (\mathbf{y}^* \cdot \mathbf{y} - \phi(\mathbf{y})) \tag{A.8}$$

Moreover, the following relationships are equivalent:

(i) $\mathbf{x}^* \in \partial\phi(\mathbf{x})$
(ii) $\mathbf{x} \in \partial\phi^*(\mathbf{x}^*)$
(iii) $\phi(\mathbf{x}) + \phi^*(\mathbf{x}^*) = \mathbf{x}^* \cdot \mathbf{x}$ (A.9)

Condition (i) is equivalent to $\mathbf{x}^* \cdot \mathbf{x} - \phi(\mathbf{x}) \geqslant \mathbf{x}^* \cdot \mathbf{y} - \phi(\mathbf{y})$. The supremum of the second term of this inequality is equal by definition to $\phi^*(\mathbf{x}^*)$ and occurs when $\mathbf{y} = \mathbf{x}$ and therefore (iii) is the same as (i). Dually, (ii) and (iii) are equivalent.

Remark 1. Under the previous assumptions, if $\phi \geqslant 0$ and $\phi(\mathbf{0}) = 0$, then (A.7) entails that $\phi^*(\mathbf{0}) = 0$. Moreover, the identity $\phi^{**} = \phi$ implies that $\phi(\mathbf{0}) = \sup_{\mathbf{y}^*\in\mathbb{X}^*}(-\phi^*(\mathbf{y}^*))$, which in turn leads to $\phi^* \geqslant 0$. Reciprocally, $\phi^* \geqslant 0$ and $\phi^*(\mathbf{0}) = 0$ entail that $\phi \geqslant 0$ and $\phi(\mathbf{0}) = 0$.

Remark 2. If ϕ^* is such that $\phi^* \geqslant 0$ and $\phi^*(\mathbf{0}) = 0$, then the normality condition (ii) implies that $\mathbf{x}^* \cdot \mathbf{x} \geqslant 0$.

Proof. Condition (ii) is equivalent to (i), with $\phi \geqslant 0$ and $\phi(\mathbf{0}) = 0$. Then, using the result of item 4, the non-negativity of $\mathbf{x}^* \cdot \mathbf{x}$ follows.

Remark 3. The conjugate $\tilde{\phi}^*$ of an arbitrary function $\tilde{\phi} : \mathbb{X} \to (-\infty, \infty]$ can still be defined by (A.7). In this case, $\tilde{\phi}^*$ is proper, convex, lower semi-continuous and is equal to the conjugated ϕ^* of $\phi = cl(\text{conv}\ \tilde{\phi})$, where ϕ is the greatest proper convex lower semi-continuous function majorized by $\tilde{\phi}$ (Rockafellar, 1969, pp. 52, 103–104).

(6) A function $\phi : \mathbb{X} \to (-\infty, \infty]$ is *positively homogeneous of order 1* if and only if

$$\forall \mathbf{y} \in \mathbb{X},\ \forall \rho \in (0, \infty), \quad \phi(\rho\mathbf{y}) = \rho\phi(\mathbf{y}) \tag{A.10}$$

The epigraph of such functions is a cone (Rockafellar, 1969, p. 30).

Given $\phi : \mathbb{X} \to (-\infty, \infty]$, the following three statements are equivalent:

(i) ϕ is proper, convex, lower semi-continuous and positively homogeneous of order 1.
(ii) The Legendre–Fenchel conjugate ϕ^* of ϕ is the indicator function of a non-empty, convex and closed set $\overline{\mathbb{E}}$, i.e.

$$\phi^*(\mathbf{y}^*) = \mathbb{I}_{\overline{\mathbb{E}}}(\mathbf{y}^*) = \begin{cases} 0 & \text{if } \mathbf{y}^* \in \overline{\mathbb{E}} \\ +\infty & \text{if } \mathbf{y}^* \in \overline{\mathbb{E}} \end{cases}$$

(iii) ϕ is the *support function* of a non-empty, convex and closed set $\overline{\mathbb{E}}$, i.e.

$$\phi(\mathbf{y}) = \mathbb{I}^*_{\overline{\mathbb{E}}}(\mathbf{y}) = \sup_{\mathbf{y}^* \in \overline{\mathbb{E}}} (\mathbf{y}^* \cdot \mathbf{y})$$

The equivalence between (i) and (ii) can be proved by showing that ϕ^* has no values other than 0 and $+\infty$ (Rockafellar, 1969, p. 114). The set where $\phi^* = 0$ is non-empty, convex and closed since ϕ is proper, convex and lower semi-continuous. The equivalence between (ii) and (iii) follows from the definition of Legendre–Fenchel transform, support functions and indicator functions.

Remark. If ϕ fulfils conditions in (i), then for any $\mathbf{x}$ where ϕ is subdifferentiable,

$$\phi(\mathbf{x}) = \phi^{**}(\mathbf{x}) = \mathbf{x}^* \cdot \mathbf{x} \quad \text{with } \mathbf{x}^* \in \partial\phi(\mathbf{x})$$

Proof. From the equivalence between (i) and (ii), the conjugated of ϕ is the indicator function of a closed convex set $\overline{\mathbb{E}}$ and $\mathbf{x}^* \in \overline{\mathbb{E}}$ since ϕ is subdifferentiable at $\mathbf{x}$ by assumption. Then, use Eq. (A.9) and recall by (ii) that $\phi^*(\mathbf{x}^*) = 0$.

(7) Let $\phi : \mathbb{X} \to (-\infty, +\infty]$ be a proper, convex, lower semi-continuous function, positively homogeneous of order 1. Then:

(i) From item (6), its conjugate ϕ^* is the indicator function of a non-empty, closed and convex set $\overline{\mathbb{E}}$. Hence, by using the definition (A.4),

$$\partial\phi^*(\mathbf{x}^*) = \partial\mathbb{I}_{\overline{\mathbb{E}}}(\mathbf{x}^*) = \begin{cases} \mathbf{0} & \text{if } \mathbf{x}^* \in \text{int}(\overline{\mathbb{E}}) \\ \mathscr{C}(\mathbf{x}^*) & \text{if } \mathbf{x}^* \in \partial\overline{\mathbb{E}} \\ \varnothing & \text{if } \mathbf{x}^* \notin \overline{\mathbb{E}} \end{cases} \tag{A.11}$$

where $\mathscr{C}(\mathbf{x}^*) = \{\mathbf{x} \in \mathbb{X} : \forall \mathbf{y}^* \in \overline{\mathbb{E}} \quad \mathbf{x} \cdot (\mathbf{y}^* - \mathbf{x}^*) \leqslant 0\}$ is the so-called *normal cone* at $\mathbf{x}^* \in \partial\overline{\mathbb{E}}$.
(ii) If in addition ϕ *does not depend on some components* $\mathbf{y}_1$ *of* $\mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2) \subset \mathbb{X} = \mathbb{X}_1 \times \mathbb{X}_2$, i.e. $\phi(\mathbf{y}) = \phi(\mathbf{y}_1, \mathbf{y}_2) = \hat{\phi}(\mathbf{y}_2)$, then the conjugated function ϕ^* can be computed as follows:

$$\phi^*(\mathbf{y}_1^*, \mathbf{y}_2^*) = \sup_{(\mathbf{y}_1, \mathbf{y}_2) \in \mathbb{X}} (\mathbf{y}_1^* \cdot \mathbf{y}_1 + \mathbf{y}_2^* \cdot \mathbf{y}_2 - \hat{\phi}(\mathbf{y}_2)) = \mathbb{I}_0(\mathbf{y}_1^*) + \sup_{\mathbf{y}_2 \in \mathbb{X}_2} (\mathbf{y}_2^* \cdot \mathbf{y}_2 - \hat{\phi}(\mathbf{y}_2)) = \mathbb{I}_0(\mathbf{y}_1^*) + \mathbb{I}_{\mathbb{E}}(\mathbf{y}_2^*) \tag{A.12}$$

The Legendre–Fenchel conjugate is the indicator function of $\mathbf{0}$ with respect to $\mathbf{y}_1^*$ plus the Legendre–Fenchel conjugate of $\hat{\phi}(\mathbf{x}_2)$, which is the indicator function of a non-empty, closed and convex set $\mathbb{E}$. Hence,

$$\mathbf{x} \in \partial\mathbb{I}_{\overline{\mathbb{E}}}(\mathbf{x}^*) \Longleftrightarrow \mathbf{x}_1 \in \mathbb{X}_1 \text{ and } \mathbf{x}_2 \in \partial\mathbb{I}_{\mathbb{E}}(\mathbf{x}_2^*)$$

In the particular case where $\mathbb{E} = \{\mathbf{y}^* \in \mathbb{X}^* \text{ such that } f(\mathbf{y}^*) \leqslant 0\}$, where f is a convex and smooth function, the normality condition at $\mathbf{y}^* = \mathbf{x}^*$, viz. $\mathbf{x} \in \partial \mathbb{I}_{\mathbb{E}}(\mathbf{x}^*)$, can be written as follows

$$\mathbf{x} = \mu \text{ grad } f(\mathbf{x}^*) \quad \text{with} \begin{cases} \mu = 0 & \text{for } f(\mathbf{x}^*) < 0 \\ \mu \geqslant 0 & \text{for } f(\mathbf{x}^*) = 0 \end{cases}$$

These two last conditions are often replaced by

$$\mu \geqslant 0, \quad f(\mathbf{x}^*) \leqslant 0, \quad \mu f(\mathbf{x}^*) = 0 \tag{A.13}$$

which are the classical loading–unloading conditions of plasticity, usually written with μ replaced by the *plastic multiplier* $\dot{\lambda}$. The dependence of f on the argument $\mathbf{x}^*$ is often omitted in order to simplify the notation. In the convex mathematical programming literature, (A.13) are known as Kuhn–Tucker conditions (see e.g. Luenberger, 1984).

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Silvano Erlicher received the Dip.-Eng. and Ph.D. degrees in Civil Engineering from the University of Trento, Italy, in 1999 and 2003, respectively. From May 2003 to April 2004, he held a post-doctoral research position at ENPC-LCPC, Paris, France. In 2004, he joined the Laboratoire d'Analyse des Matériaux et Structures (LAMI-ENPC/Institut Navier), Paris, France. His research interests include seismic engineering with special emphasis on cyclic inelastic behavior of materials and structures.

Nelly Point obtained the *agrégation* in Mathematics as student at the Ecole Normale Supérieure (ENS) in Paris, France. She received her Ph.D. degree in 1975 and her *thèse d'état* in 1989. She joined the Conservatoire National des Art et Métiers (CNAM) in 1973, where she teaches Mathematics for engineers. She develops her research activity at the Laboratoire d'Analyse des Matériaux et Identification (LAMI-ENPC/Institut Navier), Paris, France. Her research interests include applied mathematics and mechanics, in particular adhesion, fracture, plasticity and hysteresis.